The Evaluation of Integrals. A Collection of Questions

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The question of evaluation of definite integrals is one of the basic questions of elementary Mathematics.

In this talk the author will present a collection of problems that have appeared in the attempt to evaluate definite integrals. These will include some problems in Number Theory such as what is the power of 2 which divides factorials and binomial coefficients or when is the central binomial coefficient exactly divisible by 2? This connection comes from Wallis’ formula

\[
\int_0^\infty \frac{dx}{(x^2 + 1)^{n+1}} = \frac{\pi}{2^{2n+1}} \binom{2n}{n}.
\]

There are also some problems in Dynamical Systems such as describe the behavior of the map

\[
x_n = \frac{n + x_{n-1}}{1 - nx_{n-1}},
\]

with \(x_1 = 1\). It is curious that the first 4 values of \(x_n\) are integers. It has been conjectured that this never happens again. There are also examples of Taylor series: it is a remarkable fact that the coefficients in the expansion of

\[
h(x) = \sqrt{a + \sqrt{1 + x}}
\]

are related the integral

\[
\int_0^\infty \frac{dx}{(x^4 + 2ax^2 + 1)^n}.
\]

A theorem of Ramanujan gives the connection. It is a classical result that Taylor coefficients of special functions have interesting arithmetic properties. For instance, the Bernoulli numbers, appearing in the expansion of \(\tan x\), are connected to diophantine equations.
The talk will present a tour of these ideas and it will accessible to undergraduate students.