Interchanging of limit processes

Seq. and Series of Functions

<u>Def.:</u> $\forall n \in \mathbb{N}$, $f_n : E \to \mathbb{C}$

Suppose $\{f_n(x)\}$ converges $\forall x \in E$,

(1) $$f(x) := \lim_{n\to\infty} f_n(x) \quad \text{for } x \in E.$$

We say $\{f_n\}$ converges to f <u>pointwise</u>

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Similarly, if $\sum_{n=0}^{\infty} f_n(x)$ converges $\forall x \in E$,

(2) $$f(x) := \sum_{n=0}^{\infty} f_n(x).$$

We say $\sum_{n=0}^{\infty} f_n(x)$ converges to f <u>pointwise</u>.

Q.: If $\{f_n\}$ are continuous / differentiable / integrable, is the same true for the limit function?

<u>continuity:</u> $\lim_{t\to x} f(t) = f(x)$.

$$\lim_{t\to x} \lim_{n\to\infty} f_n(t) \overset{?}{=} \lim_{n\to\infty} \lim_{t\to x} f_n(t).$$

<u>Ex:</u> $$\lim_{n\to\infty} \underbrace{\lim_{m\to\infty} \frac{m}{m+n}}_{=1} = 1$$

$$\lim_{m\to\infty} \underbrace{\lim_{n\to\infty} \frac{m}{m+n}}_{=0} = 0$$

Problems: Analysis Test Bank: I2, I5, I9, I11, I21, I23.

Ex: $f_n(x) = \dfrac{x^2}{(1+x^2)^n}$

$$f(x) = \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}$$

$x = 0, \quad f_n(0) = 0 \quad \forall n \quad \Rightarrow \quad f(0) = 0$

$x \neq 0. \quad \displaystyle\sum_{n=0}^{\infty} \frac{1}{(1+x^2)^n} = \sum_{n=0}^{\infty} \frac{1}{r^n} = \frac{1}{1-r}$

$$1 - r = 1 - \frac{1}{1+x^2} = \frac{x^2}{1+x^2}$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{1}{(1+x^2)^n} = \frac{1+x^2}{x^2}$$

$$\Rightarrow \sum_{n=0}^{\infty} f_n(x) = 1 + x^2$$

$$\Rightarrow f(x) = \begin{cases} 0 & x = 0 \\ 1 + x^2 & x \neq 0 \end{cases}$$

convergent series of continuous functions that has a discontinuous sum.

Ex: $f_m(x) = \lim\limits_{n \to \infty} (\cos m! \pi x)^{2n}$

$$f_m(x) = \begin{cases} 1 & m! x \in \mathbb{Z} \\ 0 & m! x \notin \mathbb{Z} \end{cases}$$

Let $f(x) = \lim\limits_{m \to \infty} f_m(x)$.

$x \in \mathbb{Q}, \ x = \frac{p}{q}, \ p, q \in \mathbb{Z} \Rightarrow m! x = m! \frac{p}{q} \in \mathbb{Z}$ if $m \geq q \Rightarrow f(x) = 1$

$x \in \mathbb{R} / \mathbb{Q} \Rightarrow f_m(x) = 0 \ \forall m \Rightarrow f(x) = 0.$

$$f(x) = \begin{cases} 0 & x \in \mathbb{R}/\mathbb{Q} \\ 1 & x \in \mathbb{Q}. \end{cases}$$

$f(x)$ is discontinuous everywhere $\Rightarrow$ not Riemann-integrable

Ex: $f_n(x) = \dfrac{\sin nx}{\sqrt{n}}$

$$f(x) = \lim_{n\to\infty} f_n(x) = 0 \quad \Rightarrow \quad f'(x) = 0$$

$$f_n'(x) = \sqrt{n} \cos nx$$

$$\lim_{n\to\infty} f_n' \neq f', \qquad f_n'(0) = \sqrt{n} \xrightarrow[n\to\infty]{} \infty.$$

Ex: $f_n(x) = n^2 x (1-x^2)^n, \quad x \in [0,1]$

$$\lim_{n\to\infty} f_n(x) = 0 = f(x) \text{ (L'Hospital's Rule)}, \qquad \int_0^1 f(x) = 0.$$

$$\int_0^1 x(1-x^2)^n dx = \frac{1}{2n+2}, \text{ so } \int_0^1 f_n(x)dx = \frac{n^2}{2n+2} \xrightarrow[n\to\infty]{} \infty$$

limit of the integral does not equal to the integral of the limit

$$g_n(x) = n x (1-x^2)^n$$
$$\lim g_n(x) = 0 = g(x) \quad \Rightarrow \quad \int_0^1 g(x)dx = 0$$

$$\lim \int g_n(x)\,dx = \frac{n}{2n+2} \xrightarrow[n\to\infty]{} \frac{1}{2}$$

Def.: $\{f_n\}$ converges uniformly on E to f if

$$\| f_n - f \|_{\sup} \to 0$$

OR

$\forall \varepsilon > 0 \ \exists N$ s.th. $n \geq N$ $|f_n(x) - f(x)| < \varepsilon$ for $x \in E$.

Note: uniform convergence $\Rightarrow$ pointwise convergence

Similarly, $\sum f_n(x)$ <u>converges uniformly</u> on E if $\{ S_n(x) = \sum_{k=0}^{n} f_k(x) \}$ converges uniformly on E.

<u>Weierstrass M-test:</u>

$\{f_n\}$
Suppose $|f_n(x)| \le M_n$, $x \in E$.
If $\sum M_n$ converges, then $\sum f_n$ converges both uniformly and absolutely.
Moreover, $\sum |f_k(x)|$ converges uniformly on E as well.

<u>Th:</u> Suppose $f_n \to f$ uniformly on E
(or $\sum_{k=0}^{\infty} f_k \to f$ uniformly on E).

Then
1) $\{f_n\}$ seq. of continuous functions, then f is continuous on E.
($f = \sum f_k$ is continuous on E)

2) $\{f_n\}$ seq. of Riemann integrable functions on $[a,b]$ ($\mathcal{R}[a,b]$)
then $f \in \mathcal{R}[a,b]$ and
$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$$
(Similarly, $\int_a^b f = \sum_{k=0}^{\infty} \int_a^b f_k$)

<u>Th:</u> If $\{f_n\}$ is a seq. of differentiable functions on $[a,b]$, and $\exists\, x_0 \in [a,b]$ s.th. $\lim_{n\to\infty} f_n(x_0)$ exists, and $\{f_n'\}$ converges uniformly on $[a,b]$,
then $\{f_n\}$ converges uniformly on $[a,b]$ to a function f, and $f'(x) = \lim_{n\to\infty} f_n'(x)$, $x \in [a,b]$.

(Similarly, $\{f_n\}$ diff.able, $\exists x_0$ s.th. $\sum f_k(x_0)$ exists, $\sum f_n'$ converges uniformly on $[a,b]$, then $\sum f_k$ converges to f uniformly on $[a,b]$ and $f'(x) = \sum_{k=0}^{\infty} f_k'(x)$.)