# Solutions to the Problems

I2: Identify all subsets of $[0,1]$ on which $\sum_{n=0}^{\infty} x^n$ converges uniformly. Explain.

Solution:

Claim: $\sum X^n$ converges uniformly on any subset $A$ of $[0,1]$ that is held away from 1, i.e. $\sup_{x\in A} A < 1$.

Let $f_n(x) = x^n$, $M_n = m^n$ where $m = \sup_{x\in A} A$,

then $\sum M_n = \sum m^n = \frac{1}{1-m} < \infty$ for $m<1$

This proves the claim.

It remains to show: if $\sup A = 1$, then $\sum X^n$ does not converge uniformly.

$$\sup_{x\in A} |S(x) - S_n(x)| = \sup_{x\in A} \left|\sum_{k=n+1}^{\infty} x^k\right|$$

Let $\varepsilon = \frac{1}{2}$, $\tilde{X} > \sqrt[N]{\frac{3}{4}}$, then $\left|\sum_{k=N}^{\infty} \tilde{x}^k\right| > \frac{3}{4} > \frac{1}{2} = \varepsilon$

$\Rightarrow \sup_{x\in A} |S(x) - S_n(x)| > \frac{1}{2}$ $\Rightarrow$ $S_n(x)$ does not converges uniformly to $S(x)$.

$S_n(x) =$ $n$-th partial sum $= \sum_{k=0}^{n} x^k$.

Remark:

to show $S_n \to S$ unif. we have to show:

$\forall \varepsilon > 0 \ \exists N$ s.th. $|S_n - S| < \varepsilon \quad \forall n > N \quad \forall x$

So to show $S_n$ does not converge uniformly to $S$ we have to show:

$\exists \varepsilon$ s.th. $\forall N \ \exists x_N$ s.th. $|S_n(x_N) - S(x_N)| > \varepsilon \quad n > N$

In our case: $\varepsilon = \frac{1}{2}$ and $x_N = \sqrt[N]{\frac{3}{4}}$. □

I5: Prove or disprove the following two statements:

(a) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n \cos nx$ converges pointwise everywhere on $\mathbb{R}$.

(b) If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n \cos nx$ converges to a continuous function on $\mathbb{R}$.

Solution:

a) Consider a seq. $a_n = \frac{(-1)^{n+1}}{n}$, then $\sum_{n=1}^{\infty} a_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges.

Now, consider $\sum a_n \cos nx_0$ for $x_0 = \pi$,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cos n\pi = -1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} - \cdots = -\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges.}$$

This disproves the claim (a).

b) Let $M_n = |a_n|$, $f_n = a_n \cos nx$, then $|f_n(x)| \leq |a_n| = M_n$

$\sum M_n$ converges by assumption $\Rightarrow$ by W-M-test $\sum f_n$ converges uniformly.

So we have a cont. seq. of functions $\{f_n\}$ s.th. $\sum f_n$ converges uniformly, thus it converges to a continuous function on $\mathbb{R}$.

□

I9: Let $f_n : [1, \infty) \to \mathbb{R}$ be defined by $f_n(x) := \frac{n+1}{n} e^{-nx}$. Show that the series $\sum_{k=1}^{\infty} f_k$ converges uniformly to a continuous function.

Solution:

Let $M_n = \max_{x \in [1,\infty)} f_n(x) = \max_{x \in [1,\infty)} \frac{n+1}{n} e^{-nx} = \frac{n+1}{n e^n}$

If $\sum M_n$ converges, then by W-M-Test $\sum f_n$ converges unif.

To show $\sum M_n$ converges, use Ratio Test:

$$\frac{M_{n+1}}{M_n} = \frac{\frac{n+2}{n+1\, e^{n+1}}}{\frac{n+1}{n e^n}} = \frac{(n+2)\, n}{(n+1)\,(n+1)\, e} \longrightarrow \frac{1}{e} < 1$$

So by Ratio Test $\sum M_n < \infty$.

□

I.11 Let $f_n : [0,\infty) \to \mathbb{R}$ be defined by $f_n(x) := \left(\frac{x}{n}\right) e^{-\left(\frac{x}{n}\right)}$, $n \in \mathbb{N}$.

(a) Determine $f(x) = \lim_{n\to\infty} f_n(x)$. Show that the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly to $f$ on $[0,a]$ for any non-negative real number $a$. Does the seq. converge uniformly to $f$ on $[0,\infty)$? Justify your answer.

(b) Show that
$$\lim_{n\to\infty} \int_0^a f_n(x)\,dx = \int_0^a f(x)\,dx,$$
but that
$$\lim_{n\to\infty} \int_0^\infty f_n(x)\,dx \neq \int_0^\infty f(x)\,dx.$$

Solution:

(a)
$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \left(\frac{x}{n}\right) e^{-\left(\frac{x}{n}\right)} = 0 \cdot 1 = 0 =: f(x)$$

On $[0,a]$: $\sup_{0\le x\le a} \left| \frac{x}{n} e^{-\left(\frac{x}{n}\right)} \right| = \max_{0\le x\le a} \frac{x}{n e^{\frac{x}{n}}} = \frac{a}{n e^{a/n}}$

$\sup |f_n(x)| = f_n(a) \longrightarrow 0$ as $n\to\infty$

So $\sup |f_n(x) - f(x)| = f_n(a) \longrightarrow 0$ as $n \to \infty$

Hence $f_n \to f$ uniformly on $[0,a]$.

On $[0,\infty)$: Let $x_n = n$, $f_n(x_n) = \frac{n}{n} e^{-\frac{n}{n}} = e^{-1}$

$$\sup_{x\in[0,\infty)} |f_n(x_n) - f(x)| = f_n(x_n) = e^{-1}$$

Thus for any $\varepsilon < e^{-1}$ and $\forall N \in \mathbb{N}$ $\exists\, \tilde{x} \in [0,\infty)$, namely $\tilde{x} = N$ s.th. $|f_N(\tilde{x}) - f(\tilde{x})| = e^{-1} > \varepsilon$.

Hence $\{f_n\}$ does not converge uniformly on $[0,\infty)$. //

(b) $f=0 \Rightarrow \int_0^a f(x)\,dx = 0 = \int_0^\infty f(x)\,dx$

wts: $\lim_{n\to\infty} \int_0^a f_n(x)\,dx = 0$ and $\lim_{n\to\infty} \int_0^\infty f_n(x)\,dx \neq 0$

- $f_n(x) = \frac{x}{n e^{x/n}} \le \frac{a}{n e^{a/n}} \le \frac{a}{n}$ on $[0,a]$

$$0 \le \lim_{n\to\infty} \int_0^a f_n(x)\,dx \le \lim_{n\to\infty} \int_0^a \frac{a}{n}\,dx = \lim_{n\to\infty} \frac{a}{n}\cdot a = 0$$

So $0 \le \lim_{n\to\infty} \int_0^a f_n(x)\,dx \le 0$

Thus, $\lim_{n\to\infty} \int_0^a f_n(x)\,dx = 0.$

- $\int_0^\infty f_n(x)\,dx = \int_0^\infty \frac{x}{n} e^{-\left(\frac{x}{n}\right)}\,dx = \int_0^\infty \underbrace{x}_{u}\ \underbrace{\frac{1}{n} e^{-\left(\frac{x}{n}\right)}}_{v'}\,dx$

$$= \underbrace{x\left(-e^{-\left(\frac{x}{n}\right)}\right)\Big|_0^\infty}_{=0} - \int_0^\infty \left(-e^{-\frac{x}{n}}\right)dx = \left[-n e^{-\left(\frac{x}{n}\right)}\right]_0^\infty$$

$$= \underbrace{\lim_{x\to\infty}\left(-n e^{-\left(\frac{x}{n}\right)}\right)}_{=0} + n.$$

Thus, $\lim_{n\to\infty} \int_0^\infty f_n(x)\,dx = \infty.$ $\square$

I.21
Let $f_N(x) = \sum_{n=1}^{N} a_n \sin(nx)$ for $a_n, x \in \mathbb{R}$. If $\sum_{n=1}^{\infty} na_n$ converges absolutely, show that $(f_N)_{N\in\mathbb{N}}$ converges uniformly to a function $f$ on $\mathbb{R}$, and that $(f_N')_{N\in\mathbb{N}}$ converges uniformly to $f'$ on $\mathbb{R}$.

Solution:

- $f_n$ is diff-able $\forall n$

- $\forall n \in \mathbb{N}$ $|a_n| \le |n a_n| \Rightarrow \sum |a_n| \le \sum |n a_n| < \infty$
  Let $M_n = |a_n|$, and $h_n = a_n \sin(nx)$,
  then $|h_n| \le M_n$ $\forall x \in [0,\pi]$.
  By W-M-Test $f_N(x)$ converges uniformly on $[0,\pi]$.
  $f_N(x)$ is a periodic function $\Rightarrow$ converges uniformly on $\mathbb{R}$.

- Let $M_n = |n a_n|$, $h_n'(x) = n a_n \cos(nx)$, and $|h_n'| \le |na_n| = M_n$ for $x \in [0,\pi]$.
  W-M-Test implies that $f_N'(x)$ converges uniformly on $[0,\pi]$, extends to $\mathbb{R}$.

Thus $\sum a_n \sin(nx)$ converges uniformly on $\mathbb{R}$ to say $f$
and $f'(x) = \sum h_n'(x) = \sum n a \cos(nx) = \lim f_N'(x)$ $\square$

I.23: Show that a function $f(x) = e^{-x} + 2e^{-2x} + \cdots + ne^{-nx} + \cdots$ is continuous on $(0,\infty)$.

Solution:

Let $f_N(x) = \sum_{n=0}^{N} ne^{-nx}$. $\forall N$ $f_N$ is a continuous function.

Show: $\{f_N\}$ converges uniformly.
Let $a > 0$, $\forall x \in [a,\infty)$ $|f_n(x)| \le ne^{-na}$.
If $\sum ne^{-na}$ converges, then by W-M-Test $\sum ne^{-nx}$ converges uniformly on $[a,\infty)$. Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim \frac{n+1}{n} \frac{e^{na}}{e^{(n+1)a}} = \lim_{n\to\infty} \frac{n+1}{ne^{a}} < 1$

Thus $f_N(x)$ converges uniformly on $[a,\infty)$ $\forall a > 0$. $\square$