Topics in Multivariable Calculus

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Consider $\mathbb{R}^n$ with the standard Euclidean metric:

$$d(x,y) = \|y-x\|_n = \left(\sum_{i=1}^{n}(y_i-x_i)^2\right)^{1/2},$$

and let $\{e_1, e_2, \dots, e_n\}$ be the standard basis in $\mathbb{R}^n$ where $e_i = (0,0,\dots,0,1,0,\dots,0)$ with 1 at ith place. Each $f : \mathbb{R}^n \to \mathbb{R}^m$ can be writen as $f(x) = (f_1(x), \dots, f_m(x))$.

Let $U \subset \mathbb{R}^n$ be an open set. Consider a map $f : U \to \mathbb{R}^m$. We say that f is differentiable at $x \in U$ if there is a linear operator A from $\mathbb{R}^n$ to $\mathbb{R}^m$ such that

$$\lim_{h\to 0} \frac{\|f(x+h) - f(x) - Ah\|_m}{\|h\|_n} = 0 \tag{1}$$

This means that in a small neighborhood of x the function f can be approximated by the linear operator A. One can check that if such an A exists, then it is unique. In this case A is called the **Differential** of f at x and is denoted by $Df(x)$. If no such operator exists, then f is not differentiable at x.

Definition 1. The matrix of $Df(x)$ in the standard basis is called the **Derivative** of f at x, and is denoted by $f'(x)$.

We say that $f : U \to \mathbb{R}^m$ is continuously differentiable on U if the function $f' : U \to L(n,m)$ is continuous in the norm defined by $\|A\| = \max\limits_{\|h\|\le 1}\{\|Ah\|\}$. We write $f \in C^1(U, \mathbb{R}^m)$.

Theorem 2. *Let $U \subset \mathbb{R}^n$ be an open subset, and let $f : U \to V \subset \mathbb{R}^m$ be differentiable at $x \in U$, and let $g : V \to \mathbb{R}^l$ be differentiable at $y = f(x)$. Then the composite function $h : U \to \mathbb{R}^l$, defined by $h = g \circ f$ is differentiable at x, and furthermore,*

$$Dh(x) = Dg(f(x)) \circ Df(x)$$

i.e. $h'(x) = g'(y) \cdot f'(x)$, where " $\cdot$ " is matrix multiplication.

Theorem 3. *Let $f : U \to V \subset \mathbb{R}^m$ be differentiable at $x \in U$. Then for each $1 \le i \le n$, the following limit exists*

$$\lim_{t\to 0} \frac{f_i(x+te_j) - f_i(x)}{t}$$

and is equal to the ith component of $Df(x)e_j$, denoted by $\dfrac{\partial f_i}{\partial x_j}$.

Theorem 4 (Inverse Function Theorem)**.** *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable in an open set containing a, and let $\det f'(a) \neq 0$. Then there is an open set U containing a and and open set V containing $f(a)$ such that $f : U \to V$ has a continuous inverse $g = f^{-1} : V \to U$ which is also continuousely differentiable and for any $y \in V$ satisfies*

$$g'(y) = [f'(g(y))]^{-1}.$$

Definition 5. Let U be an open set in $\mathbb{R}^n$ and V be an open set in $\mathbb{R}^m$. We say that $f : U \to V$ is C^1-diffeomorphism from U to V if f is a homeomorphism, and both f and f^{-1} are continuousely differentiable.

Example 6. Show that if $\mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^m$, then $n = m$.

Theorem 7 (Implicit Function Theorem)**.** *Suppose that $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is continuously differentiable in an open set containing the point (a,b), and let $f(a,b) = 0$. Let M be a matrix*

$$\left(\frac{\partial f_i}{\partial x_j}\right) \quad 1 \le i \le m, \quad n+1 \le j \le n+m.$$

If $\det M \neq 0$*, then there is an open set* $U \subset \mathbb{R}^n$ *containing* a *and an open set* $V \subset \mathbb{R}^m$ *containing* b *such that for each* $x \in U$ *there is a unique* $y = g(x) \in V$ *that satisfies the equation* $f(x, y) = 0$*. The function* $g : U \to V$ *is continuously differentiable.*

Theorem 8 (Theorem about the Rank)**.** *Let* $U \subset \mathbb{R}^n$ *be an open set containing a point* a*. Suppose that* $f : U \to \mathbb{R}^m$ *is continuously differentiable with* $f(a) = b$*, and of constant rank* $\operatorname{rank} f'(x) = p$ *for all* $x \in U$*. Then there is an open set* $V \subset U$ *containing* a *and a diffeomorphism* $g : V \to g(V)$ *onto* $g(V)$ *containing* 0*, and an open set* $W \subset \mathbb{R}^m$ *with diffeomorphism* $h : W \to h(W)$*, with* $0 \in h(W)$*, such that the map* $\bar{f} = g^{-1} \circ f \circ h : g(V) \to h(W) \subset \mathbb{R}^m$ *is given by*

$$\bar{f}(x_1, x_2, \ldots, x_p, \ldots, x_n) = (y_1, y_2, \ldots y_p, 0, \ldots, 0). \tag{2}$$

This theorem claims that the set $f^{-1}(b)$ is $(n-p)$-dimensional subspace in $\mathbb{R}^n$ (i.e. a manifold of dimension $n - p$), and that $f(U)$ is p-dimensional subspace in $\mathbb{R}^m$ (in general, self-intersecting).

Example 9. In $\mathbb{R}^n$ consider the unit sphere S^{n-1}, given by the equation ${x_1}^2 + \cdots + {x_n}^2 - 1 = 0$. Then S^{n-1} is $(n-1)$-dimensional subspace of $\mathbb{R}^n$. The derivative $f'(x) = (2x_1, 2x_2, \ldots 2x_n)$ has rank 1 for all $x \in S^{n-1}$.

Example 10. Let $f : \mathbb{R}^n \to \mathbb{R}$ be of class C^2. Show that the non-degenerated critical points of f are isolated.

Problems

Problem 1. Check whether the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(x) = \|x\|_n$ is differentiable at 0 for $n \geq 2$. If the answer is affirmative, find $f'(0)$.

Problem 2. Give an example of a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ such that each directional derivative

$$Df(x)v = D_v f(x) = \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t}$$

exists at $x = 0$, but f is not differentiable at 0.

Problem 3. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(x, y) = \begin{cases} (x^2 + y^2) \sin \frac{1}{(x^2+y^2)^{1/2}} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Show that f is differentiable at $(0, 0)$ but partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are not continuous at $(0, 0)$.

Problem 4. If $f : \mathbb{R} \to \mathbb{R}$ satisfies $f'(a) \neq 0$ for all $a \in \mathbb{R}$, then f is $1 - 1$.

Problem 5. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (e^x \cos(y), e^x \sin(y))$. Show that $\det f'(a) \neq 0$ for all $a \in \mathbb{R}^2$, but f is not $1 - 1$.

References

1. Spivak, M., *Calculus on Manifolds.*
2. Rudin, W., *Principles of Mathematical Analysis.*
3. Richardson, L., *Advanced Calculus: An Introduction to Linear Analysis.*
4. Garrity, T., *All the Mathematics You Missed (but Need to Know for the Graduate School).*
5. Milnor, J., *Morse Theory.*
6. Analysis Test Bank for the Comprehensive Exam in Analysis at the Louisiana State University.