MÖBIUS TRANSFORMATIONS IN HYPERBOLIC SPACE

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Abstract. We wish to construct an isometric transformation that maps a hyperbolic triangle to a congruent triangle. This is accomplished by first finding the map that takes a triangle to its normal position. In two-dimensions, a triangle in normal position has one leg on the y-axis, a vertex at $(0, 1)$, and a vertex in Quadrant I. In three dimensions, a triangle in normal position has one leg on the z-axis, a vertex at $(0, 0, 1)$, and a vertex on the yz-plane. If two hyperbolic triangles are congruent, then they have the same normal position. This implies that a map between two congruent hyperbolic triangles can be constructed by transforming one triangle to the normal position and applying the inverse of the transformation that maps the other triangle to the normal position.

1. Background Information

Consider the hyperbolic plane, a geometry modelled by the upper half plane. Each point in the hyperbolic plane can be considered to be a complex number in the following way:

$$\boldsymbol{x} = (x_1, x_2) = x_1 + x_2\mathbf{i}.$$

We denote the set of all points in the hyperbolic plane as $\mathbb{H}^2$. The metric associated to the hyperbolic plane

$$\rho(\boldsymbol{x}, \boldsymbol{y}) = \operatorname{arccosh}\left(1 + \frac{\|\boldsymbol{x} - \boldsymbol{y}\|^2}{\operatorname{Im}(\boldsymbol{x})\operatorname{Im}(\boldsymbol{y})}\right)$$

where $\boldsymbol{x}$ and $\boldsymbol{y}$ are the complex representations of two points in the hyperbolic plane [2]. A geodesic is the shortest path between two points. In $\mathbb{H}^2$, geodesics are segments of vertical lines or arcs of semicircles centered on the x-axis. Our particular objects of interest in a hyperbolic space are hyperbolic triangles.

Definition 1.1. A hyperbolic triangle is a geometric object formed by three points in a hyperbolic space and the geodesics connecting them.

There are angle-preserving maps on hyperbolic space called Möbius transformations.

Definition 1.2. A Möbius transformation is a map from a hyperbolic space to itself described by

$$\phi(\boldsymbol{x}) = \frac{a\boldsymbol{x} + b}{c\boldsymbol{x} + d}$$

where $\boldsymbol{x}$ is the complex analogue of a point in hyperbolic space and a, b, c, and d are constants such that $ad - bc \neq 0$.

Note that a Möbius transformation can be encoded by a matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Fact 1.3. *Möbius transformations on $\mathbb{H}^2$ are isometries if they are encoded by a matrix with determinant one.* [1]

For this reason we are interested in the special linear group.

Definition 1.4. The special linear group of 2×2 matrices, or $SL_2(\mathbb{R})$, is the set of 2×2 real matrices with determinant one.

Note that $SL_2(\mathbb{R})$ forms a group under matrix multiplication. This is true because $SL_2(\mathbb{R})$ satisfies the group axioms:

- There exists an identity matrix in $SL_2(\mathbb{R})$.
- Matrix multiplication is associative in $SL_2(\mathbb{R})$.
- Each element of $SL_2(\mathbb{R})$ has an inverse in $SL_2(\mathbb{R})$.
- The product of any two elements of $SL_2(\mathbb{R})$ is always an element of $SL_2(\mathbb{R})$.

We will refer to a Möbius transformation in $\mathbb{H}^2$ as a fractional linear transformation if it fits the following definition.

Definition 1.5. A fractional linear transformation f_M is a Möbius transformation encoded by a matrix $M \in SL_2(\mathbb{R})$.

Note that the fractional linear transformation encoded by the identity matrix is the identity map, which is to say that for all $\boldsymbol{x}$, $f_I(\boldsymbol{x}) = \boldsymbol{x}$. The negated identity matrix, another member of $SL_2(\mathbb{R})$, also encodes the identity map.

Theorem 1.6. *The fractional linear transformation of a product of matrices is the composition of fractional linear transformations of the matrices, that is $f_{AB} = f_A \circ f_B$.*

Proof. Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

We evaluate

$$\begin{aligned} (f_A \circ f_B)(\boldsymbol{x}) &= f_A(f_B(\boldsymbol{x})) \\ &= \frac{a_{11}\frac{b_{11}\boldsymbol{x}+b_{12}}{b_{21}\boldsymbol{x}+b_{22}} + a_{12}}{a_{21}\frac{b_{11}\boldsymbol{x}+b_{12}}{b_{21}\boldsymbol{x}+b_{22}} + a_{22}} \\ &= \frac{(a_{11}b_{11} + a_{12}b_{21})\boldsymbol{x} + (a_{11}b_{12} + a_{12}b_{22})}{(a_{21}b_{11} + a_{22}b_{21})\boldsymbol{x} + (a_{21}b_{12} + a_{22}b_{22})} \\ &= f_{AB}(\boldsymbol{x}) \end{aligned}$$

□

Consider $PSL_2(\mathbb{R})$.

Definition 1.7. The projective special linear group, $PSL_2(\mathbb{R})$, is the set of 2×2 real matrices with determinant one, and a matrix and its negation are equivalent.

Note that all of the matrices in $SL_2(\mathbb{R})$ are also in $PSL_2(\mathbb{R})$, but there is not a distinction of uniqueness between a matrix and its negation. Let $M \in PSL_2(\mathbb{R})$. The equivalency classes of $PSL_2(\mathbb{R})$ are of the type $[M] = \{M, -M\}$. A proof that $PSL_2(\mathbb{R})$ forms a group can be easily obtained by ammending the reasoning used to show $SL_2(\mathbb{R})$ is a group to include equivalency classes.

Corollary 1.8. *Fractional linear transformations can be considered to be encoded by matrices in $PSL_2(\mathbb{R})$.*

Proof. Consider f_M. Using the identity maps and 1.6, we can show

$$\begin{aligned} f_M &= f_{IM} \\ &= f_I \circ f_M \\ &= f_{-I} \circ f_M \\ &= f_{-M}. \end{aligned}$$

This shows $M \equiv -M$ when encoding a fractional linear transformation. Hence, the additional condition on members of $PSL_2(\mathbb{R})$ is met. □

Corollary 1.9. *The inverse of a fractional linear transformation encoded by the matrix A is equivalent to the fractional linear transformation encoded by the matrix A^{-1}, that is $f_A^{-1} = f_{A^{-1}}$*

Proof.

$$\begin{aligned} f_A \circ f_A^{-1}(z) &= z \\ f_A \circ f_A^{-1} &= f_I \\ f_A^{-1} \circ f_A \circ f_A^{-1} &= f_A^{-1} \circ f_{AA^{-1}} \\ f_A^{-1} &= f_A^{-1} \circ f_A \circ f_{A^{-1}} \\ &= f_{A^{-1}} \end{aligned}$$

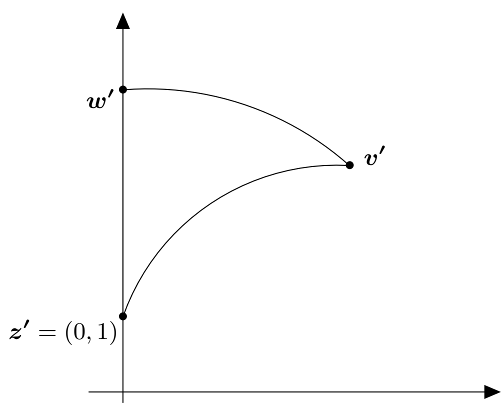
□

2. Möbius Transformations in $\mathbb{H}^2$

Let $\triangle \boldsymbol{vwz}$ be a hyperbolic triangle with counterclockwise orientation.

Definition 2.1. The normal position of $\triangle \boldsymbol{vwz}$ is the image of the triangle under an isometry that maps $\boldsymbol{z}$ to $(0, 1)$, the vertex $\boldsymbol{w}$ onto the y-axis above $\boldsymbol{z}$, and $\boldsymbol{v}$ to a point in Quadrant I.

The following is an example of a hyperbolic triangle in normal position



Note that depending on choice of vertices, a hyperbolic triangle will take one of three normal positions. Since congruent hyperbolic triangles will necessarily have the same set of normal positions, we can use the inverse of the transformation to a normal position construct a map to a congruent hyperbolic triangle.

Theorem 2.2. *There exists a fractional linear transformation that sends* $\triangle \boldsymbol{vwz}$ *to its normal position.*

We want to find a series of elementary transformations that will send $\triangle \boldsymbol{vwz}$ to its normal position.

Lemma 2.3. *The fractional linear transformation that will send $\boldsymbol{z}$ to the y-axis is encoded by*

$$M_1 = \begin{bmatrix} 1 & -z_1 \\ 0 & 1 \end{bmatrix}.$$

The fractional linear transformation encoded by M_1 is

$$f_{M_1}(\boldsymbol{x}) = x_1 + x_2 i - z_1 = (x_1 - z_1, x_2).$$

This transformation maps the vertices of the triangle in the following way:

$$\begin{aligned} f_{M_1}(\boldsymbol{v}) &= (v_1 - z_1, v_2) \\ f_{M_1}(\boldsymbol{w}) &= (w_1 - z_1, w_2) \\ f_{M_1}(\boldsymbol{z}) &= (0, z_2). \end{aligned}$$

This eliminates the real part of $\boldsymbol{z}$, leaving it on the y-axis.

A simple dilation is now needed to scale $f_{M_1}(\boldsymbol{z}) = (0, z_2)$ to the point $(0, 1)$.

Lemma 2.4. *The matrix corresponding to the fractional linear transformation that will send $f_{M_1}(\boldsymbol{z}) = (0, z_2)$ to the point $(0, 1)$ is*

$$M_2 = \frac{1}{\sqrt{z_2}} \begin{bmatrix} 1 & 0 \\ 0 & z_2 \end{bmatrix}.$$

The fractional linear transformation encoded by M_2 is

$$f_{M_2}(\boldsymbol{x}) = \frac{\frac{1}{\sqrt{z_2}}(x_1 + x_2 i)}{\frac{1}{\sqrt{z_2}} z_2} = \frac{x_1 + x_2 i}{z_2} = \left(\frac{x_1}{z_2}, \frac{x_2}{z_2} \right).$$

This transformation maps the image of the triangle under previous transformations in the following way:

$$\begin{aligned} f_{M_2} \circ f_{M_1}(\boldsymbol{v}) &= \left(\frac{v_1 - z_1}{z_2}, \frac{v_2}{z_2} \right) \\ f_{M_2} \circ f_{M_1}(\boldsymbol{w}) &= \left(\frac{w_1 - z_1}{z_2}, \frac{w_2}{z_2} \right) \\ f_{M_2} \circ f_{M_1}(\boldsymbol{z}) &= (0, 1). \end{aligned}$$

The composition of fractional linear transformations, we achieve the desired result. Note that the following matrix corresponds to the fractional linear transformation that will map $\triangle \boldsymbol{vwz}$ such that $\boldsymbol{z}$ is sent to $(0, 1)$:

$$M_1 M_2 = \frac{1}{\sqrt{z_2}} \begin{bmatrix} 1 & -z_1 \\ 0 & z_2 \end{bmatrix}.$$

Now that the image of $\boldsymbol{z}$ is $(0,1)$, we focus on the vertex $\boldsymbol{w}$. Recall, we wish to send $\boldsymbol{w}$ to a point $(0,P)$, where $P>1$.

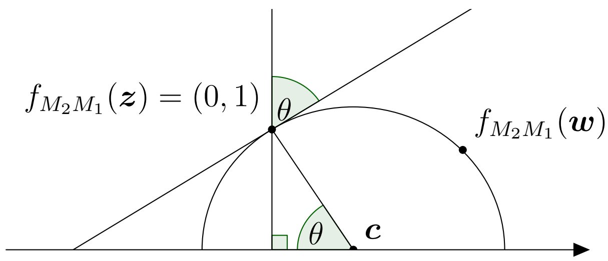
To send $f_{M_2M_1}(\boldsymbol{w}) = (\frac{w_1-z_1}{z_2}, \frac{w_2}{z_2})$ to the y-axis, we use a rotation transformation.

Lemma 2.5. *The fractional linear transformation that sends $f_{M_2M_1}(\boldsymbol{w})$ to the y-axis is encoded by*

$$K = \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$$

where θ is the directed angle from the tangent line to the geodesic between $f_{M_2M_1}(\boldsymbol{z})$ and $f_{M_2M_1}(\boldsymbol{w})$ at $f_{M_2M_1}(\boldsymbol{z})$ at the point closest to $f_{M_2M_1}(\boldsymbol{w})$ to the y-axis.

Let $\boldsymbol{c} = \left(\frac{w_2^2+(w_1-z_1)^2-z_2^2}{2(w_1-z_1)z_2}, 0\right)$ be the center of the semicircle between $f_{M_2M_1}(\boldsymbol{z})$ and $f_{M_2M_1}(\boldsymbol{w})$.



By the above figure,

$$\theta = \begin{cases} -\arctan(\frac{2(w_1-z_1)z_2}{w_2^2+(w_1-z_1)^2-z_2^2}) & \text{if } \frac{w_1-z_1}{z_2} < 0 \\ \arctan(\frac{2(w_1-z_1)z_2}{w_2^2+(w_1-z_1)^2-z_2^2}) & \text{if } \frac{w_1-z_1}{z_2} \geq 0 \end{cases}$$

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The fractional linear transformation encoded by K is

$$f_K(\boldsymbol{x}) = \frac{\cos(\frac{\theta}{2})(x_1+x_2i) - \sin\frac{\theta}{2}}{\sin(\frac{\theta}{2})(x_1+x_2i) + \cos\frac{\theta}{2}}$$

When we apply the this fractional linear transformation to $f_{M_2M_1}(\boldsymbol{z})$, we receive $(0,1)$. Applying f_K to $f_{M_2M_1}(\boldsymbol{z})$ yields a point on the y-axis.

Proof of 2.2. By lemmas 2.3,2.4, and 2.5, the fractional linear transformation that sends $\triangle\boldsymbol{vwz}$ to its normal positon is $f_{KM_2M_1}$. □

Corollary 2.6. *There exists an isometric map between any two congruent hyperbolic triangles.*

Proof. Let $\triangle\boldsymbol{uvw}$ and $\triangle\boldsymbol{xyz}$ be congruent hyperbolic triangles. Let M and N encode the transformations that send $\triangle\boldsymbol{uvw}$ and $\triangle\boldsymbol{xyz}$ to normal position, respectively. Then $f_{N^{-1}M}$ will send $\triangle \boldsymbol{uvw}$ to $\triangle \boldsymbol{xyz}$. □

3. Background Information for $\mathbb{H}^3$

Now consider hyperbolic space, a geometry modelled by the upper half space. Each point in hyperbolic space can be considered a quaternionic number in the following way:

$$\boldsymbol{x} = x_1, x_2, x_3 = x_1 + x_2\mathbf{i} + x_3\mathbf{j}.$$

Definition 3.1. The reversion operator, denoted $(\cdot)^*$, is defined such that

$$\boldsymbol{x}^* = (x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k})^* = x_1 + x_2\mathbf{i} + x_3\mathbf{j} - x_4\mathbf{k}.$$

For more information on quaternions see [3]. We denote the set of all points in hyperbolic space as $\mathbb{H}^3$. In $\mathbb{H}^3$, geodesics are segments of vertical lines or arcs of semicircles centered on the xy-plane.

We will refer to Möbius transformations on $\mathbb{H}^3$ as fractional linear transformations if they are encoded by a Vahlen matrix.

Definition 3.2. A Vahlen matrix is a 2×2 matrix of the form

$$V = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

such that following conditions are satisfied:

- a, b, c, and d are quaternions.
- ab^*, cd^*, c^*a, and d^*b are in the linear span of $\mathbb{1}$, $\mathbf{i}$, and $\mathbf{j}$.
- $\det V = ad^* - bc^*$ is a nonzero real number.

Fractional linear transformations on $\mathbb{H}^3$ are isometries.

Proposition 3.3. *Vahlen Matrices form a group under multiplication with identity element*

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and the inverse

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad * -bc*} \begin{bmatrix} d* & -b* \\ -c* & a* \end{bmatrix}$$

Since Vahlen matrices form a group, and 1.6 does not require commutitivity, 1.6 and 1.9 hold for fractional linear transformations in $\mathbb{H}^3$.

4. Möbius Transformations in $\mathbb{H}^3$

Let $\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{z} \in \mathbb{H}^3$ such that $\boldsymbol{v} = (v_1, v_2, v_3)$, $\boldsymbol{w} = (w_1, w_2, w_3)$ and $\boldsymbol{z} = (z_1, z_2, z_3)$.

Definition 4.1. The normal position of $\triangle \boldsymbol{vwz}$ is the image of the triangle under an isometry that maps $\boldsymbol{z}$ to $(0, 0, 1)$, the vertex $\boldsymbol{w}$ onto the z-axis above $\boldsymbol{z}$, and $\boldsymbol{v}$ to a point on the yz-plane.

Theorem 4.2. *There exists a fractional linear transformation that sends $\triangle \boldsymbol{vwz}$ to its normal position.*

We want to find a series of elementary transformations that will send $\triangle \boldsymbol{vwz}$ to its normal position.

Lemma 4.3. *The fractional linear transformation that will send $\boldsymbol{z}$ to $(0, 0, 1)$ is encoded by*

$$C = \begin{bmatrix} 1 & -z_1 - z_2 i \\ 0 & z_3 \end{bmatrix}$$

The fractional linear transformation encoded by C is

$$f_C(\boldsymbol{x}) = (x_1 + x_2 i + x_3 j - z_1 - z_2 i)(z_3)^{-1} = (\frac{x_1 - z_1}{z_3}, \frac{x_2 - z_2}{z_3}, \frac{x_3}{z_3}).$$

This transformation maps the vertices of the triangle in the following way:

$$\begin{aligned} f_C(\boldsymbol{v}) &= (\frac{v_1 - z_1}{z_3}, \frac{v_2 - z_2}{z_3}, \frac{v_3}{z_3}) \\ f_C(\boldsymbol{w}) &= (\frac{w_1 - z_1}{z_3}, \frac{w_2 - z_2}{z_3}, \frac{w_3}{z_3}) \\ f_C(\boldsymbol{z}) &= (0, 0, 1). \end{aligned}$$

We want to send $\boldsymbol{v}$ and $\boldsymbol{w}$ to the yz-plane.

Lemma 4.4. *Let $\phi = \arctan(\frac{w_1 - z_1}{w_2 - z_2})$. The fractional linear transformation that will send $\boldsymbol{v}$ and $\boldsymbol{w}$ to the yz-plane is encoded by*

$$B = \begin{bmatrix} -\sin(\frac{\phi}{2}) i - \cos(\frac{\phi}{2}) & 0 \\ 0 & \sin(\frac{\phi}{2}) i - \cos(\frac{\phi}{2}) \end{bmatrix}.$$

The fractional linear transformation encoded by B is

$$\begin{aligned} f_B(\boldsymbol{x}) &= -(\cos(\frac{\phi}{2}) + \sin(\frac{\phi}{2})\mathbf{i})(x_1 + x_2\mathbf{i} + x_3\mathbf{j})(-\cos(\frac{\phi}{2}) + \sin(\frac{\phi}{2})\mathbf{i})^{-1} \\ &= (x_1 \cos\phi - x_2 \sin\phi, x_1 \sin\phi + x_2 \cos\phi, x_3) \end{aligned}$$

This transformation maps the image of the triangle under previous transformations in the following way:

$$\begin{aligned} f_B \circ f_C(\boldsymbol{v}) &= \Bigg(\frac{(v_1-z_1)(w_2-z_2)-(v_2-z_2)(w_1-z_1)}{z_3\sqrt{(w_1-z_1)^2+(w_2-z_2)^2}}, \\ &\quad \frac{(v_1-z_1)(w_1-z_1)-(v_2-z_2)(w_2-z_2)}{z_3\sqrt{(w_1-z_1)^2+(w_2-z_2)^2}}, \frac{v_3}{z_3}\Bigg) \\ f_B \circ f_C(\boldsymbol{w}) &= \left(0, \frac{\sqrt{(w_1-z_1)^2+(w_2-z_2)^2}}{z_3}, \frac{w_3}{z_3}\right) \\ f_B \circ f_C(\boldsymbol{z}) &= (0,0,1). \end{aligned}$$

Now that the points $\boldsymbol{z}$,$\boldsymbol{w}$,$\boldsymbol{v}$ have been sent to the ij-plane, we wish to send to fix $\boldsymbol{z''}$ to j, send $\boldsymbol{w''}$ to the j-axis and keep $\boldsymbol{v''}$ in the ij-plane. Let $h_0 = \frac{{w'_1}^2+{w'_2}^2+{w'_3}^2-1}{2\sqrt{{w'_1}^2+{w'_2}^2}}$,$r_0 = \sqrt{1+h_0^2}$ and $r_2 = \sqrt{1+(h_0+r_0)^2}$

$$A = \begin{bmatrix} -(h_0+r_0) & ((h_0+r_0)^2-r_2^2)i \\ -i & -(h_0+r_0) \end{bmatrix}$$

Theorem 4.5. *The composition of f_A and f_{BC} fixes $\boldsymbol{z''}$ to j, sends $\boldsymbol{w''}$ to the j-axis and keeps $\boldsymbol{v'}$ to the ij-plane.*

Proof.

$$\begin{aligned} f_{ABC}(\boldsymbol{z}) &= f_A \circ f_{BC}(\boldsymbol{z}) = f_A(j) \\ &= (-(h_0+r_0)j + ((h_0+r_0)^2 - r_2^2)i)(-k-(h_0+r_0))^{-1} \\ &= (-(h_0+r_0)j + ((h_0+r_0)^2 - r_2^2)i)(\frac{-(h_0+r_0)}{1+(h_0+r_0)^2} + \frac{k}{1+(h_0+r_0)^2}) \\ &= \frac{(r_2^2-(h_0+r_0)^2-1)(h_0+r_0)i}{1+(h_0+r_0)^2} + \frac{r_2^2 j}{1+(h_0+r_0)^2} \\ &= (0,0,1) = j \\ &= \boldsymbol{z'''} = (z_1''', z_2''', z_3''') \end{aligned}$$

$$\begin{aligned} f_{ABC}(\boldsymbol{w}) &= f_A \circ f_{BC}(\boldsymbol{w}) = f_B(iw_2''+jw_3'') \\ &= (-(h_0+r_0)(iw_2''+jw_3'') + ((h_0+r_0)^2-r_2^2)i)(-i(iw_2''+jw_3'')-(h_0+r_0))^{-1} \\ &= (-(h_0+r_0)(iw_2''+jw_3'') + ((h_0+r_0)^2-r_2^2)i)(\frac{w_2''-(h_0+r_0)+w_3''k}{(w_2''-(h_0+r_0))^2+w_3''^2}) \\ &= \boldsymbol{w'''} = (0,0,w_3''') = (w_1''', w_2''', w_3''') \end{aligned}$$

$$\begin{aligned}
f_{ABC}(\boldsymbol{v}) &= f_A \circ f_{BC}(\boldsymbol{v}) = f_B(iv_2'' + jv_3'') \\
&= (-(h_0+r_0)(iv_2''+jv_3'') + ((h_0+r_0)^2 - r_2^2)i)(-i(iv_2''+jv_3'') - (h_0+r_0))^{-1} \\
&= (-(h_0+r_0)(iv_2''+jv_3'') + ((h_0+r_0)^2 - r_2^2)i)\left(\frac{v_2'' - (h_0+r_0) + v_3''k}{(v_2''-(h_0+r_0))^2 + v_3''^2}\right) \\
&= \boldsymbol{v'''} = (0, w_2''', w_3''') = (w_1''', w_2''', w_3''')
\end{aligned}$$

□

The geometric interpretation of these calculations are more easily explained in terms of the decomposition of the matrices encoded in the LFT.

Our first matrix can be decomposed to the form

$$C = C_2C_1 = \begin{bmatrix} 1 & -z_1 - z_2 i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{z_3} & 0 \\ 0 & 1 \end{bmatrix}$$

Where C_2 will translate $\boldsymbol{z}$ to the j-axis and C_1 will dilate the translated point on the j-axis to the unit vector j.

The next matrix is broken down into a composition of two planes. When these two planes are encoded into the LFT, they become a composition of two reflections, which is a rotation. Specifically this will rotate the hyperbolic triangle from the first octant to the **ij**-plane.

$$B = B_2B_1 = \begin{bmatrix} sin(\frac{\phi}{2})k + cos(\frac{\phi}{2})j & 0 \\ 0 & -sin(\frac{\phi}{2})k + cos(\frac{\phi}{2})j \end{bmatrix} \begin{bmatrix} j & 0 \\ 0 & j \end{bmatrix}$$

Once the hyperbolic triangle is in the **ij**-plane we apply another LFT in order the reach the normal position. The decomposition of the matrix encoded into the LFT that achieves this result is the product of a reflection about the **j**-axis and an inversion about a hemisphere passing through $\boldsymbol{z}$ and $\boldsymbol{w}$.

$$A = A_2A_1 = \begin{bmatrix} (h_0+r_0)j & (r_2^2 - (h_0+r_0)^2)k \\ k & (h_0+r_0)j \end{bmatrix} \begin{bmatrix} j & 0 \\ 0 & j \end{bmatrix}$$

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