Isometries of Hyperbolic Space

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Objective

Our goal involves finding isometries of hyperbolic space. That is, we wish to find isometries mapping one hyperbolic triangle onto any congruent hyperbolic triangle.

Isometries and Geodesics

Definition

An Isometry preserves distance.

If f is an isometry and ρ is a metric,

$$\rho(x, y) = \rho(f(x), f(y)).$$

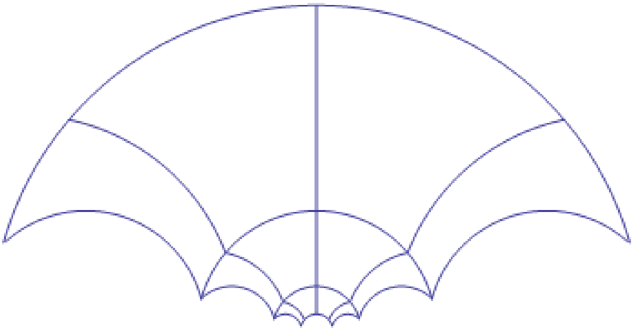
Definition

A Geodesic is a locally length-minimizing curve. Isometries map geodesics onto other geodesics.

The Hyperbolic Plane

- Similar to a Euclidian geometric 2-space, but parallel lines behave differently
- The sum of a triangle's angles is less than 180°
- Distances are based on powers of e

Hyperbolic Distances



Linear Fractional Transformations

Definition

A Linear Fractional Transformation f_M is an isometry encoded by a matrix $M \in SL_2(\mathbb{R})$. That is, for

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \; f_M(\mathbf{z}) = \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{R}$, $z \in \mathbb{C}$, and $det(M) = 1$.

Finding Geodesics: Semicircles

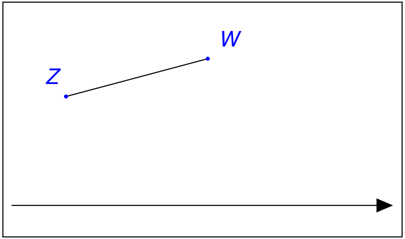
Given two points $\mathbf{z} = (z_1, z_2$ and $\mathbf{w} = (w_1, w_2)$, we wish to find the semicircle centered on the x-axis that passes through both; this will be a geodesic through the two points.

In order to perform transformations on $\mathbf{z}$ and $\mathbf{w}$, we need to find the geodesic passing through them.

Finding Geodesics: Semicircles

Using the slope and midpoint formulas, we find the slope *m* of the line through **z** and **w** to be

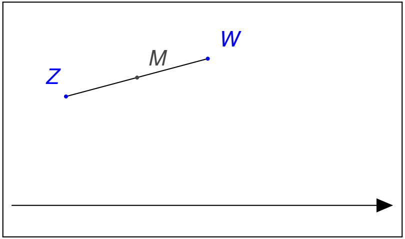
$$m = \frac{w_2 - z_2}{w_1 - z_1}$$



Finding Geodesics: Semicircles

and the midpoint M of **z** and **w** to be

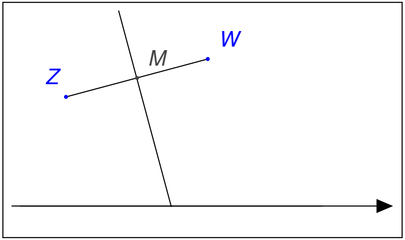
$$M = \left(\frac{z_1 + w_1}{2}, \frac{z_2 + w_2}{2} \right) = (P_1, P_2).$$



Finding Geodesics: Semicircles

Finding the perpendicular bisector of the line segment connecting **z** and **w** and taking its x-intercept, we receive the center *x* of the semicircle:

$$y - P_2 = -\frac{1}{m}(x - P_1)$$

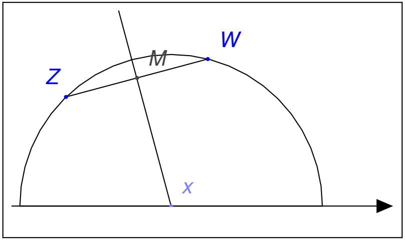


Finding Geodesics: Semicircles

Solving for x, we find that

$$x = P_2 m + P_1$$

is the center of the semicircle.



Finding Geodesics: Results

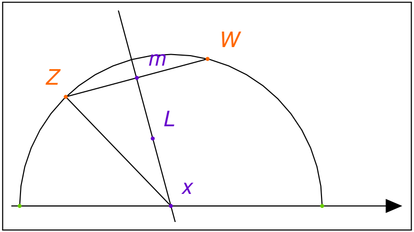
The coordinates of the center of the semicircle are then

$$(P_2 m + P_1, 0)$$

To find the radius, we take the distance between the center and **z** and receive the radius r:

$$r = \sqrt{(x - z_1)^2 + (z_2)^2}$$

Finding Geodesics: Results



This semicircle is a geodesic between the two points in hyperbolic space.

Goal

We now wish to use elementary isometries in $\mathbb{R}^2_+$ to map any hyperbolic triangle to any congruent hyperbolic triangle. This is equivalent to moving any hyperbolic triangle to a normal position; namely, $(0, 1)$ and $(0, P)$ on the y-axis.

Theorem

We can explicitly construct an isometry mapping any hyperbolic triangle to any congruent hyperbolic triangle.

Elementary Operation 1: Translating **z**

Given a vertex $\mathbf{z} = z_1 + z_2 i$ of a hyperbolic triangle, our goal is to translate this point to the point $(0, 1)$. To do this, we first translate it to the y-axis using the matrix M_1:

$$M_1 = \begin{pmatrix} 1 & -z_1 \\ 0 & 1 \end{pmatrix}$$

$$f_{M_1}(\mathbf{z}) = \frac{(z_1 + z_2 i) - z_1}{1} = z_2 \mathbf{i}.$$

Elementary Operation 2: Dilating z

Next, we use a dilation on the point $z_2\mathbf{i}$ using the matrix M_2:

$$M_2 = \begin{pmatrix} \frac{1}{\sqrt{z_2}} & 0 \\ 0 & \frac{z_2}{\sqrt{z_2}} \end{pmatrix}$$

$$f_{M_2}(z_2\mathbf{i}) = \frac{\frac{1}{\sqrt{z_2}}(z_2 i)}{\frac{z_2}{\sqrt{z_2}}} = \frac{z_2 i}{z_2} = \mathbf{i}.$$

Full Translation and Dilation of **z**

Thus the matrix M satisfying the constraints for an LFT and translating **z** to $(0, 1)$ (or $0 + i$ in complex notation) is equal to $M_2 * M_1$:

$$M = M_2 * M_1 = \begin{pmatrix} \frac{1}{\sqrt{z_2}} & 0 \\ 0 & \frac{z_2}{\sqrt{z_2}} \end{pmatrix} * \begin{pmatrix} 1 & -z_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{z_2}} & -\frac{z_1}{\sqrt{z_2}} \\ 0 & \frac{z_2}{\sqrt{z_2}} \end{pmatrix}$$

$$f_M(\mathbf{z}) = f_{M_2 M_1}(\mathbf{z}) = f_{M_2}(f_{M_1}(\mathbf{z})) = \mathbf{i}.$$

Relocating One Side of a Triangle to the Y-Axis

Next we want to send **w** to a point $(0, P)$ on the y-axis; that is,

$$\mathbf{w} = (w_1, w_2) \to (0, P), P > 1.$$

Applying M to **z**, we received $\mathbf{z}' = (0, 1)$. However, we must apply the same transformation to **w** to receive $\mathbf{w}'$:

$$f_M(\mathbf{w}) = \mathbf{w}' = \left(\frac{w_1 - z_1}{z_2}, \frac{w_2}{z_2} \right).$$

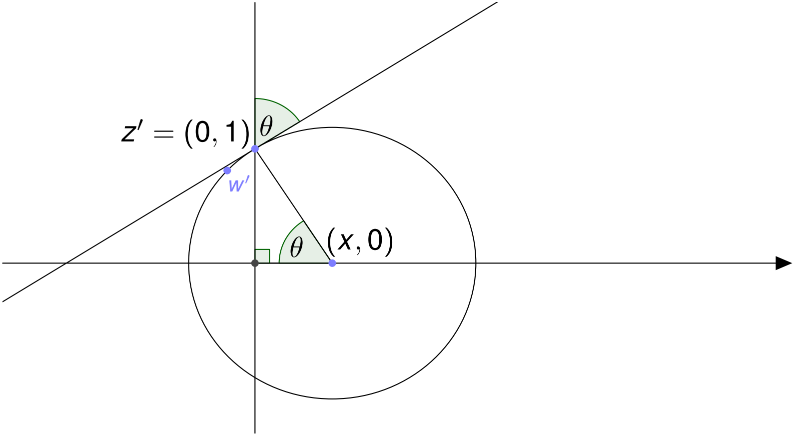
Elementary Operation 3: The Rotation Matrix K

Now we must rotate the point $\mathbf{w}'$ to the y-axis. To do this, we use a rotation matrix $K(\theta)$.

Definition

$$K(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

where θ is the directed angle between the y-axis and the line tangent to the circle passing through $\mathbf{i}$ and $\mathbf{w}'$ at $\mathbf{i}$.


$z' = (0,1)$
θ
w'
θ
$(x,0)$

Results: Applying K to $\mathbf{w}'$

By the previous diagram, $\theta = \arctan(\frac{1}{x})$, where x is the center of the circle on the x-axis passing through **z** and **w**.

When we apply the LFT $f_K(\mathbf{z}')$, we receive $(0, 1)$ again, while applying the LFT to $\mathbf{w}'$ yields

$$f_{KM_2M_1}(\mathbf{w})$$

When simplified, this gives a point $(0, P)$ on the y-axis.

Results: Applying K to $\mathbf{w}'$

$$f_{KM_2M_1}(\mathbf{w}) = \frac{\cos\left(\frac{\arctan\left(\frac{2z_2(w_1-z_1)}{w_2^2-z_2^2+(w_1-z_1)^2}\right)}{2}\right)\left(\frac{w_1-z_1}{z_2}+\frac{w_2}{z_2}i\right)+\sin\left(\frac{\arctan\left(\frac{2z_2(w_1-z_1)}{w_2^2-z_2^2+(w_1-z_1)^2}\right)}{2}\right)}{-\sin\left(\frac{\arctan\left(\frac{2z_2(w_1-z_1)}{w_2^2-z_2^2+(w_1-z_1)^2}\right)}{2}\right)\left(\frac{w_1-z_1}{z_2}+\frac{w_2}{z_2}i\right)+\cos\left(\frac{\arctan\left(\frac{2z_2(w_1-z_1)}{w_2^2-z_2^2+(w_1-z_1)^2}\right)}{2}\right)}$$

Geometric Construction

Geometrically, this construction is realized by a reflection about the y-axis and an inversion about a semicircle.

Theory Behind the $\mathbb{R}^2_+$ Case

Recall the formula used for the translation of **z**:

$$f_{M_2 M_1} = f_{M_2}(f_{M_1})$$

Theorem

The fractional linear transformation of a product of matrices is the composition of fractional linear transformations of the matrices, that is $f_{AB} = f_A \circ f_B$.

Theory Behind the $\mathbb{H}^2$ Case

Proof.

Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

Plugging A and B into f_{AB} and $f_A \circ f_B$ and simplifying, we receive that $f_{AB} = f_A \circ f_B$. □

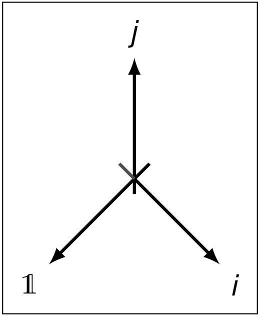
3-Dimensional Analogs

When generalized to $\mathbb{R}^3_+$:

- Reflections across lines become reflections across planes
- Inversions about semicircles become inversions about hemispheres
- Matrices in $SL_2(\mathbb{R})$ become Vahlen matrices in $M_2(\mathbb{H})$
- Rotations become rotations about a vertical axis

Quaternions

The quaternions are effectively an extension of the complex numbers in which i, j, and k are all distinct roots of -1. $\mathbb{R}^3_+$ is the span of $\mathbb{1}$, i, and j.



Goal

Given three vertices of a hyperbolic triangle **z**, **w** and **v**, we wish to send **z** to the unit vector $\mathbf{j} = (0, 0, 1)$, **w** to a point above **j** on the **j**-axis, and **v** to the $\mathbb{1}$**j**-plane.

Elementary Operation 1: Translation

We first translate the hyperbolic triangle by applying an LFT using a matrix N such that **z** is sent to the **j**-axis.

$$N = \begin{pmatrix} 1 & -z_1 - z_2 i \\ 0 & 1 \end{pmatrix}$$

$$f_N(\mathbf{z}) = \frac{(z_1 + z_2 i + z_3 j) + (-z_1 - z_2 i)}{1} = z_3 \mathbf{j}$$

Elementary Operation 2: Dilation

After the triangle is translated, we apply an LFT using a matrix *A* such that **z** is dilated from the **j**-axis to the unit vector **j**.

$$A = \begin{pmatrix} \frac{1}{z_3} & 0 \\ 0 & 1 \end{pmatrix}$$

$$f_A(z_3\mathbf{j}) = \frac{\frac{1}{z_3} * z_3 j}{1} = \mathbf{j}$$

Translating **w** and **v**

Applying f_{AN} to $\mathbf{z}$, we receive $\mathbf{z}' = \mathbf{j}$. However, we must now apply f_A and f_N to $\mathbf{w}$ and $\mathbf{v}$. Doing this, we receive

$$\mathbf{w'} = \left(\frac{w_1 - z_1}{z_3}, \frac{w_2 - z_2}{z_3}, \frac{w_3}{z_3}\right)$$

$$\mathbf{v'} = \left(\frac{v_1 - z_1}{z_3}, \frac{v_2 - z_2}{z_3}, \frac{v_3}{z_3}\right)$$

Elementary Operation 3: Rotation

The points $\mathbf{w}'$ and $\mathbf{v}'$ must now be rotated to the **ij**-plane. Let V be a plane that makes an angle $\frac{\phi}{2}$ with the $\mathbb{1}$**j**-plane.

$$B = B_2 B_1 = \begin{pmatrix} \sin(\frac{\phi}{2})k + \cos(\frac{\phi}{2})j & 0 \\ 0 & -\sin(\frac{\phi}{2})k + \cos(\frac{\phi}{2})j \end{pmatrix} \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix}$$

Elementary Operation 3: Rotation

f_B fixes **z** at **j** and yields **z**$''$. By calculation, we see that

$$\mathbf{w''} = (0, \sqrt{{w'_1}^2 + {w'_2}^2}, w'_3)$$

$$\mathbf{v''} = (0, \sqrt{{v'_1}^2 + {v'_2}^2}, v'_3)$$

which both lie on the **ij**-plane.

A Final Reflection and Inversion

Finally, we need a matrix which will fix $\mathbf{z}''$, send $\mathbf{w}''$ to the $\mathbf{j}$-axis, and keep $\mathbf{v}''$ in the $\mathbf{ij}$-plane. The matrix C satisfies these conditions:

$$C = C_2 C_1 = \begin{pmatrix} (h_0 + r_0)j & (r_2^2 - (h_0 + r_0)^2)k \\ k & (h_0 + r_0)j \end{pmatrix} \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix}$$

Results

Through the composition of the matrices C, B, A, and N, we now have an LFT analagous to the $\mathbb{R}^2_+$ case which will fix **z**, send **w** to the **j**-axis, and send **v** to the **ij**-plane. Thus the LFT

$$f_{CBAN}(\mathbf{w})$$

yields a point $(0, 0, P)$ on the **j**-axis.

For More Information...

A more detailed explanation of the $\mathbb{R}^3_+$ case, as well as proofs for the theory used in both the $\mathbb{R}^2_+$ and $\mathbb{R}^3_+$ cases, can be found in our paper.

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