Section 5.4 The Definite Integral

# Topic 1: Approximating Areas by Left and Right Sums

In this section, we will introduce the **definite integral** which is used to compute area, probabilities, average values of functions, future values of continuous income streams, and many other quantities.

The definite integral is used to find areas where a standard geometric area formula does not apply directly. We will approximate these areas using rectangles. We place a **left rectangle** on each subinterval, that is a rectangle whose base is the subinterval and whose height is the value of the function at the left endpoint of the subinterval. Summing the areas of the left rectangles is called a **left sum** denoted , where *n* denotes the number of rectangles into which the interval is broken. If the function is increasing, the left sum underestimates the area. If the function is decreasing, the left sum overestimates the area.

By the same reasoning, we could place a **right rectangle** on each subinterval, that is a rectangle whose base is the subinterval and whose height is the value of the function at the right endpoint of the subinterval. Summing the areas of the right rectangles is called a **right sum** denoted , where *n* denotes the number of rectangles into which the interval is broken. If the function is increasing, the right sum overestimates the area. If the function is decreasing, the right sum underestimates the area.

**Theorem: Limits of Left and Right Sums**

If  and is either increasing on the interval  or decreasing on , then its left and right sums approach the same real number as .

# Topic 2: The Definite Integral as a Limit of Sums

**Summation Notation**

Let a function *f* be defined on the interval . We partition  into *n* subintervals of equal length  , . Then, using summation notation, we have the following:

**Left sum: **

**Right sum: **

**Riemann sum**: ****

In a **Riemann Sum,** each  is required to belong to the subinterval . Left and Right sums are special cases of Riemann sums in which  is the left endpoint or right endpoint, respectively, of the subinterval.

**Theorem: Limit of Riemann Sums**

If *f* is a continuous function on the interval , then the Riemann sums for *f* on  approach a real number limit *I* as .

**Definite Integral**

Let *f* be a continuous function on . The limit *I* of Riemann sums for *f* on  is called the definite integral of *f* from *a* to *b* and is denoted as

.

The **integrand** is , the **lower limit of integration** is *a* and the **upper limit of integration** is *b.*  Because the area is always positive, the definite integral represents the cumulative sum of the signed areas between the graph of *f* and the *x-*axis from  to .

**Topic 3: Properties of the Definite Integral**

Let *f* and *g* be integrable functions on an interval that contains *a*, *b*, and *c*.





, for any constant *k*



