Telephone Numbers and the Umbral Calculus

April Chow[1] Iris Gray[1]
Lawrence Mouillé[2] James Pickett[3]

[1]Louisiana State University
Baton Rouge, LA

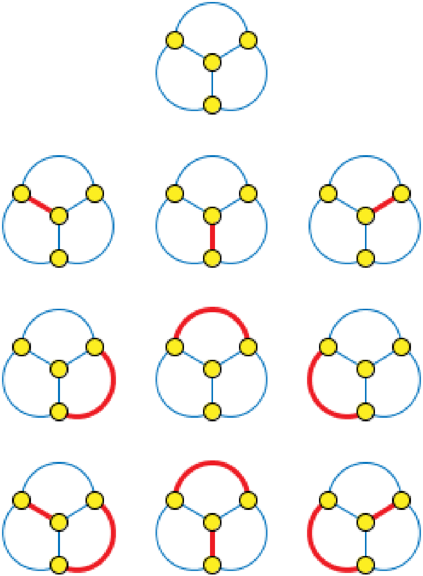
[2]University of Louisiana at Lafayette
Lafayette, LA

[3]University of South Alabama
Mobile, AL

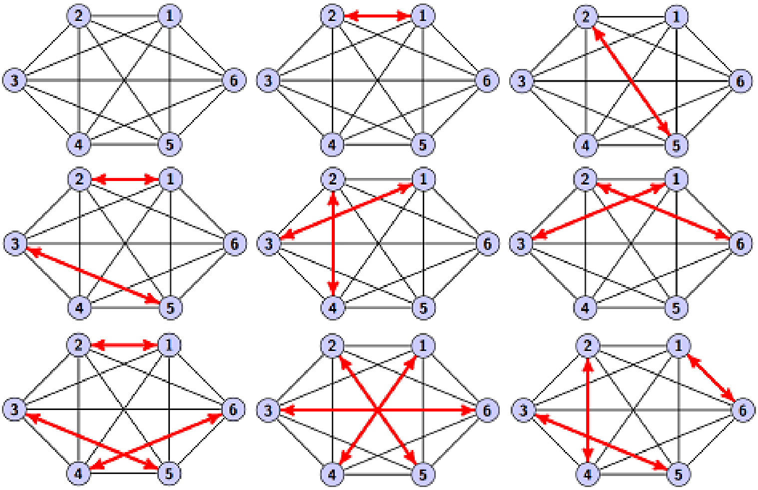
SMILE @ LSU 2013, July 12, 2013

t_4 Possible Conversations

t_4 Possible Conversations



t_6 Possible Conversations



There are 76 ways to have these conversations.

Calculation of t_0 to t_{10}

# of users	Ways of Calling
0	1
1	1
2	2
3	4
4	10
5	26
6	76
7	232
8	764
9	2620
10	9496

Recursion Relation of t_n

The telephone numbers satisfy the following recursion:

$$t_n = t_{n-1} + (n-1)t_{n-2},$$

where t_0 and t_1 are both equal to 1.

Recursion Relation of t_n

The telephone numbers satisfy the following recursion:

$$t_n = t_{n-1} + (n-1)t_{n-2},$$

where t_0 and t_1 are both equal to 1.

Proof.

Either user 1 is making a call or user 1 is not making a call.

The telephone numbers satisfy the following recursion:

$$t_n = t_{n-1} + (n-1)t_{n-2},$$

where t_0 and t_1 are both equal to 1.

Proof.

Either user 1 is making a call or user 1 is not making a call.

- If user 1 is not making a call, there are t_{n-1} ways that the network of $n-1$ remaining users can be configured. Hence, the t_{n-1} term.

The telephone numbers satisfy the following recursion:

$$t_n = t_{n-1} + (n-1)t_{n-2},$$

where t_0 and t_1 are both equal to 1.

Proof.

Either user 1 is making a call or user 1 is not making a call.

- If user 1 is not making a call, there are t_{n-1} ways that the network of $n-1$ remaining users can be configured. Hence, the t_{n-1} term.
- If user 1 is making a call, we may choose who he is calling $n-1$ different ways. In each $n-1$ case, there are $n-2$ users remaining. Hence, the $(n-1)t_{n-2}$ term.

$\square$

Theorem

$$t_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{2^k k!(n-2k)!}$$

An exponential generating function, or EGF of a_n, a sequence, can be defined as the following power series:

$$\sum_{k=0}^{\infty} \frac{x^n}{n!} a_n$$

An exponential generating function, or EGF of a_n, a sequence, can be defined as the following power series:

$$\sum_{k=0}^{\infty} \frac{x^n}{n!} a_n$$

Theorem

The EGF of t_n is

$$T(x) = \sum_{k=0}^{\infty} \frac{t_n}{n!} x^n = e^{x+\frac{x^2}{2}}.$$

Proof.

Now, recall that $t_{n+1} = t_n + nt_{n-1}$. In terms of the summation:

$$\sum_{n=1}^{\infty} \frac{t_{n+1}}{n!} x^n = \sum_{n=1}^{\infty} \frac{t_n}{n!} x^n + \sum_{n=1}^{\infty} \frac{n\, t_{n-1}}{n!} x^n$$

Proof.

Now, recall that $t_{n+1} = t_n + nt_{n-1}$. In terms of the summation:

$$\sum_{n=1}^{\infty} \frac{t_{n+1}}{n!} x^n = \sum_{n=1}^{\infty} \frac{t_n}{n!} x^n + \sum_{n=1}^{\infty} \frac{n\, t_{n-1}}{n!} x^n$$

Proof.

Now, recall that $t_{n+1} = t_n + nt_{n-1}$. In terms of the summation:

$$\sum_{n=1}^{\infty} \frac{t_{n+1}}{n!} x^n = \sum_{n=1}^{\infty} \frac{t_n}{n!} x^n + \sum_{n=1}^{\infty} \frac{n\, t_{n-1}}{n!} x^n$$

- $\displaystyle\sum_{n=1}^{\infty} \frac{t_{n+1}}{n!} x^n = T'(x) - 1.$

Proof.

Now, recall that $t_{n+1} = t_n + nt_{n-1}$. In terms of the summation:

$$\sum_{n=1}^{\infty} \frac{t_{n+1}}{n!} x^n = \sum_{n=1}^{\infty} \frac{t_n}{n!} x^n + \sum_{n=1}^{\infty} \frac{n\, t_{n-1}}{n!} x^n$$

- $\displaystyle\sum_{n=1}^{\infty} \frac{t_{n+1}}{n!} x^n = T'(x) - 1.$
- $\displaystyle\sum_{n=1}^{\infty} \frac{t_n}{n!} x^n = T(x) - 1.$

Proof.

Now, recall that $t_{n+1} = t_n + nt_{n-1}$. In terms of the summation:

$$\sum_{n=1}^{\infty} \frac{t_{n+1}}{n!} x^n = \sum_{n=1}^{\infty} \frac{t_n}{n!} x^n + \sum_{n=1}^{\infty} \frac{n\, t_{n-1}}{n!} x^n$$

- $\displaystyle\sum_{n=1}^{\infty} \frac{t_{n+1}}{n!} x^n = T'(x) - 1.$
- $\displaystyle\sum_{n=1}^{\infty} \frac{t_n}{n!} x^n = T(x) - 1.$
- $\displaystyle\sum_{n=1}^{\infty} \frac{n\, t_{n-1}}{n!} x^n = xT'(x).$

Proof (Cont.)

Solving for $T(x)$:

$$T'(x) - 1 = T(x) - 1 + xT'(x).$$

Proof (Cont.)

Solving for $T(x)$:

$$T'(x) - 1 = T(x) - 1 + xT'(x).$$

Now,

$$T'(x) = (1 + x)T(x),$$

where $T(0) = 1$

$$\implies T(x) = e^{x+\frac{x^2}{2}}$$

$\square$

The set of Hermite polynomials is a polynomial sequence.

The set of Hermite polynomials is a polynomial sequence.

Applications of Hermite Polynomials:

- Probability;

The set of Hermite polynomials is a polynomial sequence.

Applications of Hermite Polynomials:

- Probability;
- Numerical Analysis;

The set of Hermite polynomials is a polynomial sequence.

Applications of Hermite Polynomials:

- Probability;
- Numerical Analysis;
- Combinatorics;

The set of Hermite polynomials is a polynomial sequence.

Applications of Hermite Polynomials:

- Probability;
- Numerical Analysis;
- Combinatorics;
- Physics.

Definition

$$H_n(u) = n! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{k!\,(n-2k)!}(2u)^{n-2k}$$

n	$H_n(u)$
0	1
1	$2u$
2	$2(-1+2u^2)$
3	$6(-2u+\frac{4u^3}{3})$
4	$24(\frac{1}{2}-2u^2+\frac{2u^4}{3})$
5	$120(u-\frac{4u^3}{3}+\frac{4u^5}{15})$
6	$720(-\frac{1}{6}+u^2-\frac{2u^4}{3}+\frac{4u^6}{45})$
7	$5040(-\frac{u}{3}+\frac{2u^3}{3}-\frac{4u^5}{15}+\frac{8u^7}{315})$
8	$40320(\frac{1}{24}-\frac{u^2}{3}+\frac{u^4}{3}-\frac{4u^6}{45}+\frac{2u^8}{315})$
9	$362880(\frac{u}{12}-\frac{2u^3}{9}+\frac{2u^5}{15}-\frac{8u^7}{315}+\frac{4u^9}{2835})$
10	$3628800(-\frac{1}{120}+\frac{u^2}{12}-\frac{u^4}{9}+\frac{2u^6}{45}-\frac{2u^8}{315}+\frac{4u^{10}}{14175})$

Observe

$$\left(\frac{i}{\sqrt{2}}\right)^0 H_0\left(\frac{-i}{\sqrt{2}}\right) = 1 = t_0$$

Observe

$$\left(\frac{i}{\sqrt{2}}\right)^0 H_0\left(\frac{-i}{\sqrt{2}}\right) = 1 = t_0$$

$$\left(\frac{i}{\sqrt{2}}\right)^1 H_1\left(\frac{-i}{\sqrt{2}}\right) = 1 = t_1$$

Observe

$$\left(\frac{i}{\sqrt{2}}\right)^0 H_0\left(\frac{-i}{\sqrt{2}}\right) = 1 = t_0$$

$$\left(\frac{i}{\sqrt{2}}\right)^1 H_1\left(\frac{-i}{\sqrt{2}}\right) = 1 = t_1$$

$$\left(\frac{i}{\sqrt{2}}\right)^2 H_2\left(\frac{-i}{\sqrt{2}}\right) = 2 = t_2$$

Observe

$$\left(\frac{i}{\sqrt{2}}\right)^0 H_0\left(\frac{-i}{\sqrt{2}}\right) = 1 = t_0$$

$$\left(\frac{i}{\sqrt{2}}\right)^1 H_1\left(\frac{-i}{\sqrt{2}}\right) = 1 = t_1$$

$$\left(\frac{i}{\sqrt{2}}\right)^2 H_2\left(\frac{-i}{\sqrt{2}}\right) = 2 = t_2$$

$$\left(\frac{i}{\sqrt{2}}\right)^3 H_3\left(\frac{-i}{\sqrt{2}}\right) = 4 = t_3$$

Theorem

$$\left(\frac{i}{\sqrt{2}}\right)^n H_n\left(\frac{-i}{\sqrt{2}}\right) = t_n$$

Proof.

$$\left(\frac{i}{\sqrt{2}}\right)^n H_n\left(\frac{-i}{\sqrt{2}}\right) = \frac{i^n}{(\sqrt{2})^n} n! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{k!(n-2k)!} \left(2\left(\frac{-i}{\sqrt{2}}\right)\right)^{n-2k}$$

Proof.

$$\left(\frac{i}{\sqrt{2}}\right)^n H_n\left(\frac{-i}{\sqrt{2}}\right) = \frac{i^n}{(\sqrt{2})^n} n! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{k!(n-2k)!}\left(2\left(\frac{-i}{\sqrt{2}}\right)\right)^{n-2k}$$

$$= \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{2^k k!(n-2k)!}$$

Proof.

$$\left(\frac{i}{\sqrt{2}}\right)^n H_n\left(\frac{-i}{\sqrt{2}}\right) = \frac{i^n}{(\sqrt{2})^n} n! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{k!(n-2k)!}\left(2\left(\frac{-i}{\sqrt{2}}\right)\right)^{n-2k}$$

$$= \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{2^k k!(n-2k)!}$$

$$= t_n$$

□

Suppose a_n is given and we define another sequence

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k$$

Suppose a_n is given and we define another sequence

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k$$

We wish to solve for a_n in terms of b_k, and will use the umbrae.

$$B^n = \sum_{k=0}^{n} \binom{n}{k} A^k$$

$$B^n = \sum_{k=0}^{n} \binom{n}{k} A^k$$

$$B^n = \sum_{k=0}^{n} \binom{n}{k} A^k$$

$$\implies B^n = (1 + A)^n$$

$$B^n = \sum_{k=0}^{n} \binom{n}{k} A^k$$

$$\implies B^n = (1 + A)^n$$

$$\implies B = 1 + A$$

$$B^n = \sum_{k=0}^{n} \binom{n}{k} A^k$$

$$\implies B^n = (1 + A)^n$$

$$\implies B = 1 + A$$

$$\implies A = B - 1$$

$$B^n = \sum_{k=0}^{n} \binom{n}{k} A^k$$

$$\implies B^n = (1 + A)^n$$

$$\implies B = 1 + A$$

$$\implies A = B - 1$$

$$\implies A^n = (B - 1)^n$$

$$B^n = \sum_{k=0}^{n} \binom{n}{k} A^k$$

$$\implies B^n = (1 + A)^n$$

$$\implies B = 1 + A$$

$$\implies A = B - 1$$

$$\implies A^n = (B - 1)^n$$

$$\implies A^n = \sum_{k=0}^{n} \binom{n}{k} B^k (-1)^{n-k}$$

$$B^n = \sum_{k=0}^{n} \binom{n}{k} A^k$$

$$\implies B^n = (1 + A)^n$$

$$\implies B = 1 + A$$

$$\implies A = B - 1$$

$$\implies A^n = (B - 1)^n$$

$$\implies A^n = \sum_{k=0}^{n} \binom{n}{k} B^k (-1)^{n-k}$$

$$\implies a_n = \sum_{k=0}^{n} \binom{n}{k} b_k (-1)^{n-k}$$

Consider the vector space V with the basis $\{1, A, A^2, \dots\}$.

Consider the vector space V with the basis $\{1, A, A^2, \dots\}$.

Vectors in V are polynomials of the form

$$p(A) = \sum_{k=0}^{d} c_k A^k$$

Consider the vector space V with the basis $\{1, A, A^2, \dots\}$.

Vectors in V are polynomials of the form

$$p(A) = \sum_{k=0}^{d} c_k A^k$$

Define the linear functional $L : V \to \mathbb{R}$ such that $L(A^n) = a_n$.

Consider the vector space V with the basis $\{1, A, A^2, \dots\}$.

Vectors in V are polynomials of the form

$$p(A) = \sum_{k=0}^{d} c_k A^k$$

Define the linear functional $L : V \to \mathbb{R}$ such that $L(A^n) = a_n$.

Then,

$$L(p(A)) = L(\sum_{k=0}^{d} c_k A^k) = \sum_{k=0}^{d} c_k L(A^k)$$

Consider the vector space V with the basis $\{1, A, A^2, \dots\}$.

Vectors in V are polynomials of the form

$$p(A) = \sum_{k=0}^{d} c_k A^k$$

Define the linear functional $L : V \to \mathbb{R}$ such that $L(A^n) = a_n$.

Then,

$$L(p(A)) = L(\sum_{k=0}^{d} c_k A^k) = \sum_{k=0}^{d} c_k L(A^k)$$

- A is the umbra of a_n.

Consider the vector space V with the basis $\{1, A, A^2, \dots\}$.

Vectors in V are polynomials of the form

$$p(A) = \sum_{k=0}^{d} c_k A^k$$

Define the linear functional $L : V \to \mathbb{R}$ such that $L(A^n) = a_n$.

Then,

$$L(p(A)) = L(\sum_{k=0}^{d} c_k A^k) = \sum_{k=0}^{d} c_k L(A^k)$$

- A is the umbra of a_n.
- Many formulas are easier to derive if we change a_n to A^n for some variable A. After some algebraic manipulation , we change A^n back to a_n.

1. We have $b_n = \sum_{k=0}^{n} \binom{n}{k} a_k$, and want to solve for a_n in terms of b_k.
2. Recall: $L(A^n) = a_n$ and $L(B^n) = b_n$

1. We have $b_n = \sum_{k=0}^{n} \binom{n}{k} a_k$, and want to solve for a_n in terms of b_k.
2. Recall: $L(A^n) = a_n$ and $L(B^n) = b_n$

Applying these two we get, $L(B^n) = \sum_{k=0}^{n} \binom{n}{k} L(A^k)$

1. We have $b_n = \sum_{k=0}^{n} \binom{n}{k} a_k$, and want to solve for a_n in terms of b_k.
2. Recall: $L(A^n) = a_n$ and $L(B^n) = b_n$

Applying these two we get, $L(B^n) = \sum_{k=0}^{n} \binom{n}{k} L(A^k)$

$$= L\left(\sum_{k=0}^{n} \binom{n}{k} A^k\right)$$

1. We have $b_n = \sum_{k=0}^{n} \binom{n}{k} a_k$, and want to solve for a_n in terms of b_k.
2. Recall: $L(A^n) = a_n$ and $L(B^n) = b_n$

Applying these two we get, $L(B^n) = \sum_{k=0}^{n} \binom{n}{k} L(A^k)$

$$= L\left(\sum_{k=0}^{n} \binom{n}{k} A^k \right) = L\Big((1 + A)^n \Big)$$

1. We determined $L(B^n) = L[(1+A)^n]$
2. Let p(B) be a polynomial in B.

1. We determined $L(B^n) = L[(1 + A)^n]$
2. Let p(B) be a polynomial in B.

$$L(p(B)) = L\left[\sum_{k=0}^{d} c_k B^k\right]$$

1. We determined $L(B^n) = L[(1 + A)^n]$
2. Let p(B) be a polynomial in B.

$$L(p(B)) = L\left[\sum_{k=0}^{d} c_k B^k\right] = \sum_{k=0}^{d} c_k L(B^k)$$

1. We determined $L(B^n) = L[(1+A)^n]$
2. Let p(B) be a polynomial in B.

$$L(p(B)) = L\left[\sum_{k=0}^{d} c_k B^k\right] = \sum_{k=0}^{d} c_k L(B^k) = \sum_{k=0}^{d} c_k L[(1+A)^k]$$

1. We determined $L(B^n) = L[(1+A)^n]$
2. Let p(B) be a polynomial in B.

$$L(p(B)) = L\left[\sum_{k=0}^{d} c_k B^k\right] = \sum_{k=0}^{d} c_k L(B^k) = \sum_{k=0}^{d} c_k L[(1+A)^k]$$
$$= L(p(1+A))$$

$L(p(B)) = L(p(1 + A))$ for any polynomial p.

$L(p(B)) = L(p(1 + A))$ for any polynomial p.

$p(B) = (B - 1)^n$

$L(p(B)) = L(p(1 + A))$ for any polynomial p.

$p(B) = (B - 1)^n$

$$L(B - 1)^n = L((1 + A - 1)^n) = L(A^n)$$

$L(p(B)) = L(p(1 + A))$ for any polynomial p.

$p(B) = (B - 1)^n$

$$L(B - 1)^n = L((1 + A - 1)^n) = L(A^n)$$

$$L\left(\sum_{k=0}^{n} \binom{n}{k} B^k (-1)^{n-k}\right) = L(A^n)$$

$L(p(B)) = L(p(1+A))$ for any polynomial p.

$p(B) = (B-1)^n$

$$L(B-1)^n = L((1+A-1)^n) = L(A^n)$$

$$L\Big(\sum_{k=0}^{n}\binom{n}{k}B^k(-1)^{n-k}\Big) = L(A^n)$$

$$\sum_{k=0}^{n}\binom{n}{k}b_k(-1)^{n-k} = a_n$$

Theorem

$$F(u,x) = \sum_{n=0}^{\infty} \frac{H_n(u)}{n!}\, x^n = e^{2ux - x^2}$$

Theorem

$$F(u,x) = \sum_{n=0}^{\infty} \frac{H_n(u)}{n!} x^n = e^{2ux - x^2}$$

- Now we derive the EGF of the Hermite polynomials using umbral methods

Theorem

$$F(u, x) = \sum_{n=0}^{\infty} \frac{H_n(u)}{n!} x^n = e^{2ux - x^2}$$

- Now we derive the EGF of the Hermite polynomials using umbral methods
- If we define the umbra M such that $L(M^{2n+1}) = 0$ and $L(M^{2n}) = \frac{(-1)^n (2n)!}{n!}$ then $H_n(u) = L[(M + 2u)^n]$

Theorem

$$F(u,x) = \sum_{n=0}^{\infty} \frac{H_n(u)}{n!} x^n = e^{2ux - x^2}$$

- Now we derive the EGF of the Hermite polynomials using umbral methods
- If we define the umbra M such that $L(M^{2n+1}) = 0$ and $L(M^{2n}) = \frac{(-1)^n (2n)!}{n!}$ then $H_n(u) = L[(M + 2u)^n]$
- Then we have,

$$\sum_{n=0}^{\infty} \frac{H_n(u)}{n!} x^n = L(e^{(M+2u)x}) = e^{2ux} L(e^{Mx}) = e^{2ux} e^{-x^2} = e^{2ux - x^2}$$

- Let's take a look at the EGFs t_n and $H_n(u)$, they are

$$F(u, x') = \sum_{n=0}^{\infty} \frac{H_n(u)}{n!} (x')^n = e^{2ux' - (x')^2}$$

and,

$$T(x) = \sum_{n=0}^{\infty} \frac{t_n}{n!} x^n = e^{x + \frac{x^2}{2}}$$

- Let's take a look at the EGFs t_n and $H_n(u)$, they are

$$F(u, x') = \sum_{n=0}^{\infty} \frac{H_n(u)}{n!}(x')^n = e^{2ux' - (x')^2}$$

and,

$$T(x) = \sum_{n=0}^{\infty} \frac{t_n}{n!} x^n = e^{x + \frac{x^2}{2}}$$

- Notice any similarities? Choosing u and x' for $F(u, x')$ appropriately we can achieve equality. Those choices would be, $u = \frac{-i}{\sqrt{2}}$ and $x' = \frac{ix}{\sqrt{2}}$

- Let's take a look at the EGFs t_n and $H_n(u)$, they are

$$F(u, x') = \sum_{n=0}^{\infty} \frac{H_n(u)}{n!}(x')^n = e^{2ux'-(x')^2}$$

and,

$$T(x) = \sum_{n=0}^{\infty} \frac{t_n}{n!} x^n = e^{x+\frac{x^2}{2}}$$

- Notice any similarities? Choosing u and x' for $F(u, x')$ appropriately we can achieve equality. Those choices would be, $u = \frac{-i}{\sqrt{2}}$ and $x' = \frac{ix}{\sqrt{2}}$

- Proof.

$$F\left(\frac{-i}{\sqrt{2}}, \frac{ix}{\sqrt{2}}\right) = e^{2\left(\frac{-i}{\sqrt{2}}\right)\left(\frac{ix}{\sqrt{2}}\right)-\left(\frac{ix}{\sqrt{2}}\right)^2} = e^{x+\frac{1}{2}x^2} = T(x)$$

$\square$

- Writing the EGFs in their power series representations with $F\left(\frac{-i}{\sqrt{2}}, \frac{ix}{\sqrt{2}}\right) = T(x)$ we have

- Writing the EGFs in their power series representations with $F\left(\frac{-i}{\sqrt{2}}, \frac{ix}{\sqrt{2}}\right) = T(x)$ we have
- $$\sum_{n=0}^{\infty} \frac{H_n\left(\frac{-i}{\sqrt{2}}\right)}{n!} \left(\frac{ix}{\sqrt{2}}\right)^n = \sum_{n=0}^{\infty} \frac{t_n}{n!} x^n.$$

  Hence, for all n

  $$\frac{H_n\left(\frac{-i}{\sqrt{2}}\right)}{n!} \left(\frac{ix}{\sqrt{2}}\right)^n = \frac{t_n}{n!} x^n.$$

  Therefore,

  $$\left(\frac{i}{\sqrt{2}}\right)^n H_n\left(\frac{-i}{\sqrt{2}}\right) = t_n.$$

Theorem

$$\sum_{n=0}^{\infty} H_{3n}(u)\,\frac{x^n}{n!} = \frac{e^{8v^3x+144v^4x^2}}{(1+48ux)^{1/4}} \sum_{n=0}^{\infty} \frac{(-1)^n\,(6n)!}{(3n)!\,(1+48ux)^{3n/2}}\,\frac{x^{2n}}{(2n)!},$$

where $v = (\sqrt{1+48ux}-1)/(24x)$.

Theorem

$$\sum_{n=0}^{\infty} H_{3n}(u)\,\frac{x^n}{n!} = \frac{e^{8v^3x+144v^4x^2}}{(1+48ux)^{1/4}} \sum_{n=0}^{\infty} \frac{(-1)^n\,(6n)!}{(3n)!\,(1+48ux)^{3n/2}}\,\frac{x^{2n}}{(2n)!},$$

where $v = (\sqrt{1+48ux}-1)/(24x)$.

Theorem

$$\sum_{n=0}^{\infty} H_{3n}(u)\,\frac{x^n}{n!} = \frac{e^{8v^3x+144v^4x^2}}{(1+48ux)^{1/4}} \sum_{n=0}^{\infty} \frac{(-1)^n\,(6n)!}{(3n)!\,(1+48ux)^{3n/2}}\,\frac{x^{2n}}{(2n)!},$$

where $v = (\sqrt{1+48ux}-1)/(24x)$.

Theorem

$$\sum_{n=0}^{\infty} H_{3n}(u)\,\frac{x^n}{n!} = \frac{e^{8v^3x+144v^4x^2}}{(1+48ux)^{1/4}} \sum_{n=0}^{\infty} \frac{(-1)^n\,(6n)!}{(3n)!\,(1+48ux)^{3n/2}}\,\frac{x^{2n}}{(2n)!},$$

where $v = (\sqrt{1+48ux}-1)/(24x)$.

Theorem

$$\sum_{n=0}^{\infty} H_{3n}(u)\,\frac{x^n}{n!} = \frac{e^{8v^3x+144v^4x^2}}{(1+48ux)^{1/4}} \sum_{n=0}^{\infty} \frac{(-1)^n\,(6n)!}{(3n)!\,(1+48ux)^{3n/2}}\,\frac{x^{2n}}{(2n)!},$$

where $v = (\sqrt{1+48ux}-1)/(24x)$.

Theorem

$$\sum_{n=0}^{\infty} H_{3n}(u)\,\frac{x^n}{n!} = \frac{e^{8v^3x+144v^4x^2}}{(1+48ux)^{1/4}} \sum_{n=0}^{\infty} \frac{(-1)^n\,(6n)!}{(3n)!\,(1+48ux)^{3n/2}}\,\frac{x^{2n}}{(2n)!},$$

where $v = (\sqrt{1+48ux}-1)/(24x)$.

Lemma (1)

$$\sum_{n=0}^{\infty} H_{3n}(u)\,\frac{x^n}{n!} = \frac{e^{8u^3x-(\beta+z)^2+\phi}}{(1+48ux)^{1/4}} L\Big(e^{M^3\alpha/(1-48\alpha\beta)^{3/4}}\Big),$$

where $\alpha = \frac{x}{(1+24xu)^{3/2}}$, $\beta = \frac{12xu^2}{\sqrt{1+24xu}}$, z is a solution of $12\alpha z^2 + (24\alpha\beta - 1)z + 12\alpha\beta^2 = 0$, and $\phi = 2\beta z - 8\alpha\beta^3 - 24\alpha\beta^2 z + +2z^2 - 24\alpha\beta z^2 - 8\alpha z^3$.

Lemma (1)

$$\sum_{n=0}^{\infty} H_{3n}(u)\,\frac{x^n}{n!} = \frac{e^{8u^3x-(\beta+z)^2+\phi}}{(1+48ux)^{1/4}} L\left(e^{M^3\alpha/(1-48\alpha\beta)^{3/4}}\right).$$

Proof.

For any positive integer ω,

$$\sum_{n=0}^{\infty} H_{\omega n}(u)\,\frac{x^n}{n!} = L\left(e^{(M+2u)^{\omega}x}\right).$$

Lemma (1)

$$\sum_{n=0}^{\infty} H_{3n}(u)\,\frac{x^n}{n!} = \frac{e^{8u^3x-(\beta+z)^2+\phi}}{(1+48ux)^{1/4}} L\left(e^{M^3\alpha/(1-48\alpha\beta)^{3/4}}\right).$$

Proof.

For any positive integer ω,

$$\sum_{n=0}^{\infty} H_{\omega n}(u)\,\frac{x^n}{n!} = L\left(e^{(M+2u)^{\omega}x}\right).$$

So,

$$\sum_{n=0}^{\infty} H_{3n}(u)\,\frac{x^n}{n!} = L\left(e^{(M+2u)^{3}x}\right)$$

$$= e^{8u^3x} L\left(e^{6uxM^2} e^{xM^3+12u^2xM}\right).$$

Lemma (1)

$$\sum_{n=0}^{\infty} H_{3n}(u)\,\frac{x^n}{n!} = \frac{e^{8u^3x-(\beta+z)^2+\phi}}{(1+48ux)^{1/4}} L\left(e^{M^3\alpha/(1-48\alpha\beta)^{3/4}}\right).$$

Proof (Cont.)

If $f(M)$ is a power series and γ is a function not dependent on M, then

$$L(e^{\gamma M^2} f(M)) = \frac{1}{\sqrt{1+4\gamma}} L\left(f\left(\frac{M}{\sqrt{1+4\gamma}}\right)\right).$$

Lemma (1)

$$\sum_{n=0}^{\infty} H_{3n}(u)\,\frac{x^n}{n!} = \frac{e^{8u^3x-(\beta+z)^2+\phi}}{(1+48ux)^{1/4}} L\left(e^{M^3\alpha/(1-48\alpha\beta)^{3/4}}\right).$$

Proof (Cont.)

If $f(M)$ is a power series and γ is a function not dependent on M, then

$$L(e^{\gamma M^2} f(M)) = \frac{1}{\sqrt{1+4\gamma}} L\left(f\left(\frac{M}{\sqrt{1+4\gamma}}\right)\right).$$

Thus,

$$\sum_{n=0}^{\infty} H_{3n}(u)\,\frac{x^n}{n!} = e^{8u^3x} L(e^{6ux M^2} e^{xM^3+12xu^2M})$$
$$= \frac{e^{8u^3x}}{\sqrt{1+24ux}} L\left(e^{\frac{x}{(1+24ux)^{3/2}}M^3+\frac{12u^2x}{\sqrt{1+24ux}}M}\right).$$

Lemma (1)

$$\sum_{n=0}^{\infty} H_{3n}(u)\,\frac{x^n}{n!} = \frac{e^{8u^3x-(\beta+z)^2+\phi}}{(1+48ux)^{1/4}} L\Big(e^{M^3\alpha/(1-48\alpha\beta)^{3/4}}\Big).$$

Proof (Cont.)

For α and β functions not dependent on M,

$$L(e^{\alpha M^3+\beta M}) = \frac{e^{-(\beta+z)^2+\phi}}{\sqrt{1-24\alpha(\beta+z)}}\, L\Big(e^{\frac{M^3\alpha}{(1-24\alpha(\beta+z))^{3/2}}}\Big),$$

where $\phi = 2\beta z - 8\alpha\beta^3 - 24\alpha\beta^2 z + 2z^2 - 24\alpha\beta z^2 - 8\alpha z^3$, and z is a solution of $12\alpha z^2 + (24\alpha\beta - 1)z + 12\alpha\beta^2 = 0$.

Lemma (1)

$$\sum_{n=0}^{\infty} H_{3n}(u)\,\frac{x^n}{n!} = \frac{e^{8u^3x-(\beta+z)^2+\phi}}{(1+48ux)^{1/4}} L\Big(e^{M^3\alpha/(1-48\alpha\beta)^{3/4}}\Big).$$

Proof (Cont.)

$$L(e^{\alpha M^3+\beta M}) = \frac{e^{-(\beta+z)^2+\phi}}{\sqrt{1-24\alpha(\beta+z)}} L\Big(e^{\frac{M^3\alpha}{(1-24\alpha(\beta+z))^{3/2}}}\Big).$$

Lemma (1)

$$\sum_{n=0}^{\infty} H_{3n}(u)\,\frac{x^n}{n!} = \frac{e^{8u^3x-(\beta+z)^2+\phi}}{(1+48ux)^{1/4}} L\left(e^{M^3\alpha/(1-48\alpha\beta)^{3/4}}\right).$$

Proof (Cont.)

$$L(e^{\alpha M^3+\beta M}) = \frac{e^{-(\beta+z)^2+\phi}}{\sqrt{1-24\alpha(\beta+z)}} L\left(e^{\frac{M^3\alpha}{(1-24\alpha(\beta+z))^{3/2}}}\right).$$

Hence,

$$\sum_{n=0}^{\infty} H_{3n}(u)\,\frac{x^n}{n!} = \frac{e^{8u^3x}}{\sqrt{1+24ux}} L\left(e^{\frac{x}{(1+24ux)^{3/2}}M^3+\frac{12u^2x}{\sqrt{1+24ux}}M}\right)$$

$$= \frac{e^{8u^3x-(\beta+z)^2+\phi}}{\sqrt{1+24ux}\sqrt{1-24\alpha(\beta+z)}} L\left(e^{\frac{M^3\alpha}{(1-24\alpha(\beta+z))^{3/2}}}\right).$$

Lemma (1)

$$\sum_{n=0}^{\infty} H_{3n}(u) \frac{x^n}{n!} = \frac{e^{8u^3x-(\beta+z)^2+\phi}}{(1+48ux)^{1/4}} L\left(e^{M^3\alpha/(1-48\alpha\beta)^{3/4}}\right).$$

Proof (Cont.)

Substituting $\alpha = \frac{x}{(1+24xu)^{3/2}}$ and $\beta = \frac{12xu^2}{\sqrt{1+24xu}}$,

Lemma (1)

$$\sum_{n=0}^{\infty} H_{3n}(u)\,\frac{x^n}{n!} = \frac{e^{8u^3x-(\beta+z)^2+\phi}}{(1+48ux)^{1/4}} L\left(e^{M^3\alpha/(1-48\alpha\beta)^{3/4}}\right).$$

Proof (Cont.)

Substituting $\alpha = \frac{x}{(1+24xu)^{3/2}}$ and $\beta = \frac{12xu^2}{\sqrt{1+24xu}}$, we have

$$\sum_{n=0}^{\infty} H_{3n}(u)\,\frac{x^n}{n!} = \frac{e^{8u^3x-(\beta+z)^2+\phi}}{\sqrt{1+24ux}\sqrt{1-24\alpha(\beta+z)}} L\left(e^{\frac{M^3\alpha}{(1-24\alpha(\beta+z))^{3/2}}}\right)$$

$$= \frac{e^{8u^3x-(\beta+z)^2+\phi}}{(1+48ux)^{1/4}} L\left(e^{M^3\alpha/(1-48\alpha\beta)^{3/4}}\right).$$

□

Theorem

$$\sum_{n=0}^{\infty} H_{3n}(u)\,\frac{x^n}{n!} = \frac{e^{8v^3x+144v^4x^2}}{(1+48ux)^{1/4}} \sum_{n=0}^{\infty} \frac{(-1)^n\,(6n)!}{(3n)!\,(1+48ux)^{3n/2}}\,\frac{x^{2n}}{(2n)!},$$

where $v = (\sqrt{1+48ux}-1)/(24x)$.

Theorem

$$\sum_{n=0}^{\infty} H_{3n}(u)\,\frac{x^n}{n!} = \frac{e^{8v^3x+144v^4x^2}}{(1+48ux)^{1/4}} \sum_{n=0}^{\infty} \frac{(-1)^n\,(6n)!}{(3n)!\,(1+48ux)^{3n/2}}\,\frac{x^{2n}}{(2n)!},$$

where $v = (\sqrt{1+48ux}-1)/(24x)$.

Lemma (1)

$$\sum_{n=0}^{\infty} H_{3n}(u)\,\frac{x^n}{n!} = \frac{e^{8u^3x-(\beta+z)^2+\phi}}{(1+48ux)^{1/4}} L\left(e^{M^3\alpha/(1-48\alpha\beta)^{3/4}}\right),$$

Theorem

$$\sum_{n=0}^{\infty} H_{3n}(u)\,\frac{x^n}{n!} = \frac{e^{8v^3x+144v^4x^2}}{(1+48ux)^{1/4}} \sum_{n=0}^{\infty} \frac{(-1)^n\,(6n)!}{(3n)!\,(1+48ux)^{3n/2}}\,\frac{x^{2n}}{(2n)!},$$

where $v = (\sqrt{1+48ux}-1)/(24x)$.

Lemma (1)

$$\sum_{n=0}^{\infty} H_{3n}(u)\,\frac{x^n}{n!} = \frac{e^{8u^3x-(\beta+z)^2+\phi}}{(1+48ux)^{1/4}} L\left(e^{M^3\alpha/(1-48\alpha\beta)^{3/4}}\right),$$

Lemma (2)

$$L\left(e^{(M^3\alpha)/(1-48\alpha\beta)^{3/4}}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n\,(6n)!}{(3n)!\,(1+48ux)^{3n/2}}\,\frac{x^{2n}}{(2n)!}.$$

Lemma (2)

$$L\left(e^{(M^3\alpha)/(1-48\alpha\beta)^{3/4}}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n\,(6n)!}{(3n)!\,(1+48ux)^{3n/2}}\,\frac{x^{2n}}{(2n)!}.$$

Proof.

Using the series expansion of e^x,

Lemma (2)

$$L\left(e^{(M^3\alpha)/(1-48\alpha\beta)^{3/4}}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n\,(6n)!}{(3n)!\,(1+48ux)^{3n/2}}\,\frac{x^{2n}}{(2n)!}.$$

Proof.

Using the series expansion of e^x, we have

$$\begin{aligned} L\left(e^{(M^3\alpha)/(1-48\alpha\beta)^{3/4}}\right) &= L\left(\sum_{n=0}^{\infty}\frac{1}{n!}\cdot\left(\frac{M^3\alpha}{(1-48\alpha\beta)^{3/4}}\right)^n\right) \\ &= L\left(\sum_{n=0}^{\infty} M^{3n}\frac{\alpha^n}{n!\,(1-48\alpha\beta)^{3n/4}}\right). \end{aligned}$$

Lemma (2)

$$L\left(e^{(M^3\alpha)/(1-48\alpha\beta)^{3/4}}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n\,(6n)!}{(3n)!\,(1+48ux)^{3n/2}}\,\frac{x^{2n}}{(2n)!}.$$

Proof (Cont.)

Since L is linear over umbrae,

Lemma (2)

$$L\left(e^{(M^3\alpha)/(1-48\alpha\beta)^{3/4}}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n\,(6n)!}{(3n)!\,(1+48ux)^{3n/2}}\,\frac{x^{2n}}{(2n)!}.$$

Proof (Cont.)

Since L is linear over umbrae,

$$\begin{aligned} L\left(e^{(M^3\alpha)/(1-48\alpha\beta)^{3/4}}\right) &= L\left(\sum_{n=0}^{\infty} M^{3n}\frac{\alpha^n}{n!\,(1-48\alpha\beta)^{3n/4}}\right) \\ &= \sum_{n=0}^{\infty} L\left(M^{3n}\right)\frac{\alpha^n}{n!\,(1-48\alpha\beta)^{3n/4}}. \end{aligned}$$

Lemma (2)

$$L\left(e^{(M^3\alpha)/(1-48\alpha\beta)^{3/4}}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n\,(6n)!}{(3n)!\,(1+48ux)^{3n/2}}\,\frac{x^{2n}}{(2n)!}.$$

Proof (Cont.)

Since $L(M^{2n+1})=0$ and $L(M^{2n})=\frac{(-1)^n(2n)!}{n!}$,

Lemma (2)

$$L\left(e^{(M^3\alpha)/(1-48\alpha\beta)^{3/4}}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n\,(6n)!}{(3n)!\,(1+48ux)^{3n/2}}\,\frac{x^{2n}}{(2n)!}.$$

Proof (Cont.)

Since $L(M^{2n+1}) = 0$ and $L(M^{2n}) = \frac{(-1)^n(2n)!}{n!}$,

$$\begin{aligned} L\left(e^{(M^3\alpha)/(1-48\alpha\beta)^{3/4}}\right) &= \sum_{n=0}^{\infty} L(M^{3n})\frac{\alpha^n}{n!\,(1-48\alpha\beta)^{3n/4}} \\ &= \sum_{n=0}^{\infty} L(M^{6n})\frac{\alpha^{2n}}{(2n)!\,(1-48\alpha\beta)^{3n/2}} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^{3n}(6n)!}{(3n)!}\cdot\frac{\alpha^{2n}}{(2n)!(1-48\alpha\beta)^{3n/2}}. \end{aligned}$$

Lemma (2)

$$L\left(e^{(M^3\alpha)/(1-48\alpha\beta)^{3/4}}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n\,(6n)!}{(3n)!\,(1+48ux)^{3n/2}}\,\frac{x^{2n}}{(2n)!}.$$

Proof (Cont.)

Substituting for α and β,

Lemma (2)

$$L\left(e^{(M^3\alpha)/(1-48\alpha\beta)^{3/4}}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n\,(6n)!}{(3n)!\,(1+48ux)^{3n/2}}\,\frac{x^{2n}}{(2n)!}.$$

Proof (Cont.)

Substituting for α and β, we have

$$\begin{aligned} L\left(e^{(M^3\alpha)/(1-48\alpha\beta)^{3/4}}\right) &= \sum_{n=0}^{\infty} \frac{(-1)^{3n}(6n)!}{(3n)!}\cdot\frac{\alpha^{2n}}{(2n)!(1-48\alpha\beta)^{3n/2}} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n\,(6n)!}{(3n)!\,(1+48ux)^{3n/2}}\,\frac{x^{2n}}{(2n)!}. \end{aligned}$$

$\square$

Theorem

$$\sum_{n=0}^{\infty} H_{3n}(u)\,\frac{x^n}{n!} = \frac{e^{8v^3x+144v^4x^2}}{(1+48ux)^{1/4}} \sum_{n=0}^{\infty} \frac{(-1)^n\,(6n)!}{(3n)!\,(1+48ux)^{3n/2}}\,\frac{x^{2n}}{(2n)!},$$

where $v = (\sqrt{1+48ux}-1)/(24x)$.

Theorem

$$\sum_{n=0}^{\infty} H_{3n}(u)\,\frac{x^n}{n!} = \frac{e^{8v^3x+144v^4x^2}}{(1+48ux)^{1/4}} \sum_{n=0}^{\infty} \frac{(-1)^n\,(6n)!}{(3n)!\,(1+48ux)^{3n/2}}\,\frac{x^{2n}}{(2n)!},$$

where $v = (\sqrt{1+48ux}-1)/(24x)$.

Lemma (1 & 2)

$$\sum_{n=0}^{\infty} H_{3n}(u)\,\frac{x^n}{n!} = \frac{e^{8u^3x-(\beta+z)^2+\phi}}{(1+48ux)^{1/4}} \sum_{n=0}^{\infty} \frac{(-1)^n\,(6n)!}{(3n)!\,(1+48ux)^{3n/2}}\,\frac{x^{2n}}{(2n)!}.$$

Theorem

$$\sum_{n=0}^{\infty} H_{3n}(u)\,\frac{x^n}{n!} = \frac{e^{8v^3x+144v^4x^2}}{(1+48ux)^{1/4}} \sum_{n=0}^{\infty} \frac{(-1)^n\,(6n)!}{(3n)!\,(1+48ux)^{3n/2}}\,\frac{x^{2n}}{(2n)!},$$

where $v = (\sqrt{1+48ux}-1)/(24x)$.

Lemma (1 & 2)

$$\sum_{n=0}^{\infty} H_{3n}(u)\,\frac{x^n}{n!} = \frac{e^{8u^3x-(\beta+z)^2+\phi}}{(1+48ux)^{1/4}} \sum_{n=0}^{\infty} \frac{(-1)^n\,(6n)!}{(3n)!\,(1+48ux)^{3n/2}}\,\frac{x^{2n}}{(2n)!}.$$

Lemma (3)

$$e^{8u^3x-(\beta+z)^2+\phi} = e^{8v^3x+144v^4x^2},$$

where $v = (\sqrt{1+48ux}-1)/(24x)$.

Lemma (3)

$$e^{8u^3x-(\beta+z)^2+\phi} = e^{8v^3x+144v^4x^2},$$

where $v = (\sqrt{1+48ux}-1)/(24x)$.

Proof.

Substituting for β, z, and ϕ using Mathematica,

Lemma (3)

$$e^{8u^3x-(\beta+z)^2+\phi} = e^{8v^3x+144v^4x^2},$$

where $v = (\sqrt{1+48ux}-1)/(24x)$.

Proof.

Substituting for β, z, and ϕ using Mathematica, we have

$$\begin{aligned} 8xu^3-(\beta+z)^2+\phi &= \frac{1+72xu+864x^2u^2-(1+48xu)^{\frac{3}{2}}}{864x^2} \\ &= 8v^3x+144v^4x^2 \end{aligned}$$

where $v = (\sqrt{1+48ux}-1)/(24x)$.

Lemma (3)

$$e^{8u^3x-(\beta+z)^2+\phi} = e^{8v^3x+144v^4x^2},$$

where $v = (\sqrt{1+48ux}-1)/(24x)$.

Proof (Cont.)

Since

$$8xu^3-(\beta+z)^2+\phi = 8v^3x+144v^4x^2,$$

Lemma (3)

$$e^{8u^3x-(\beta+z)^2+\phi} = e^{8v^3x+144v^4x^2},$$

where $v = (\sqrt{1+48ux}-1)/(24x)$.

Proof (Cont.)

Since

$$8xu^3 - (\beta+z)^2 + \phi = 8v^3x + 144v^4x^2,$$

$$e^{8u^3x-(\beta+z)^2+\phi} = e^{8v^3x+144v^4x^2}$$

$\square$

Lemma (1)

$$\sum_{n=0}^{\infty} H_{3n}(u)\,\frac{x^n}{n!} = \frac{e^{8u^3x-(\beta+z)^2+\phi}}{(1+48ux)^{1/4}} L\left(e^{M^3\alpha/(1-48\alpha\beta)^{3/4}}\right),$$

Lemma (1)

$$\sum_{n=0}^{\infty} H_{3n}(u)\,\frac{x^n}{n!} = \frac{e^{8u^3x-(\beta+z)^2+\phi}}{(1+48ux)^{1/4}} L\left(e^{M^3\alpha/(1-48\alpha\beta)^{3/4}}\right),$$

Lemma (2)

$$L\left(e^{(M^3\alpha)/(1-48\alpha\beta)^{3/4}}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n\,(6n)!}{(3n)!\,(1+48ux)^{3n/2}}\,\frac{x^{2n}}{(2n)!}.$$

Lemma (1)

$$\sum_{n=0}^{\infty} H_{3n}(u)\,\frac{x^n}{n!} = \frac{e^{8u^3x-(\beta+z)^2+\phi}}{(1+48ux)^{1/4}} L\left(e^{M^3\alpha/(1-48\alpha\beta)^{3/4}}\right),$$

Lemma (2)

$$L\left(e^{(M^3\alpha)/(1-48\alpha\beta)^{3/4}}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n\,(6n)!}{(3n)!\,(1+48ux)^{3n/2}}\,\frac{x^{2n}}{(2n)!}.$$

Lemma (3)

$$e^{8u^3x-(\beta+z)^2+\phi} = e^{8v^3x+144v^4x^2},$$

where $v = (\sqrt{1+48ux}-1)/(24x)$.

Theorem

$$\sum_{n=0}^{\infty} H_{3n}(u)\,\frac{x^n}{n!} = \frac{e^{8v^3x+144v^4x^2}}{(1+48ux)^{1/4}} \sum_{n=0}^{\infty} \frac{(-1)^n\,(6n)!}{(3n)!\,(1+48ux)^{3n/2}}\,\frac{x^{2n}}{(2n)!},$$

where $v = (\sqrt{1+48ux}-1)/(24x)$.

Recall that

$$\left(\frac{i}{\sqrt{2}}\right)^n H_n\left(\frac{-i}{\sqrt{2}}\right) = t_n$$

Recall that

$$\left(\frac{i}{\sqrt{2}}\right)^n H_n\left(\frac{-i}{\sqrt{2}}\right) = t_n$$

Corollary

$$\sum_{n=0}^{\infty} t_{3n} \frac{x^n}{n!} = \frac{e^{-4w^3x\sqrt{2}-72w^4x^2}}{(1+24x)^{1/4}} \sum_{n=0}^{\infty} \frac{(6n)!}{(3n)!\,(1+24x)^{3n/2}} \frac{x^{2n}}{(2n)!\,2^n},$$

where

$$w = \frac{\sqrt{2}(\sqrt{1+24x}-1)}{24x}.$$

References

I.M. Gessel.
Applications of the Classical Umbral Calculus.
Algebra univers., **49** (2003), 397–434.

V. De Angelis.
Class Lecture, *The Umbral Calculus*.
Louisiana State University, Baton Rouge, LA.
Summer 2013.

Acknowledgements

Ben Warren and Dr. Valerio De Angelis