Constructing Zero Divisor Graphs

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Outline

Joint work with:

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THE PROJECT

To study the various definitions of zero divisor graphs associated to rings, namely those by I. Beck [B] and D. Anderson & P. Livingston [AL].

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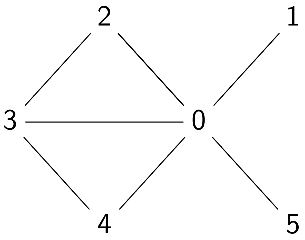
- Construct the [AL] graph $\Gamma(R)$ for $R = (\mathbb{Z}_n)$ up to order n, for $n \leq 100$.
- Observe behavior and patterns of each graph.

Relations Between Rings and Graphs

Ring

Graph

$\mathbb{Z}/6\mathbb{Z}$



A **ring** R is a set together with two binary operations $+$ and $\cdot$ (called addition and multiplication) satisfying the following axioms:

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A. R is an Abelian group under $+$; i.e.,
B. For any a, b in R, ab is in R. *(closure of multiplication)*
C. For any a, b, c in R, $a(bc) = (ab)c$. *(associativity of multiplication)*
D. For any a, b, c in R, $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$. *(distributive property)*

Example

Let R be the integers with arithmetic modulo 6. The elements of R are $\{0, 1, 2, 3, 4, 5\}$, and the operations of addition and multiplication are detailed in the tables below:

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+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

*	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

Zero Divisor Graph: The zero divisor graph of a commutative ring R with identity is a simple graph whose set of vertices consists of all elements of the ring, with an edge defined between a and b if and only if $ab = 0$.

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Unit: A unit is a non-zero element u in a commutative ring with identity such that there is another non-zero element v of the ring satisfying $uv = 1$.

Complete Graph: A graph is complete if every vertex in the graph is adjacent to every other vertex.

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Complete Bipartite Graph: A complete bipartite graph is a graph whose vertex set can be partitioned into 2 disjoint subset u_i and v_j such each u_i is adjacent to every v_j, but no two u_i's are adjacent and no two v_j's are adjacent.

Definition: [I. Beck, 1988] The (original) **zero divisor graph** of a ring R is a simple graph whose set of vertices consists of all elements of the ring, with an edge defined between a and b if and only if $ab = 0$. It will be denoted by $G(R)$.

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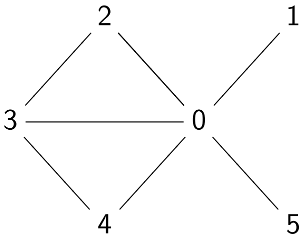
Example $R = \mathbb{Z}_6$ $\quad V(R) = \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$



$G(\mathbb{Z}_6)$

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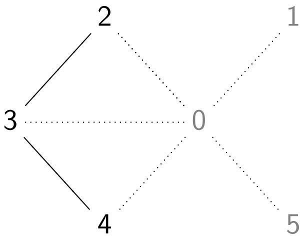
Example $R = \mathbb{Z}_6$ $\qquad V(R) = Z^*(\mathbb{Z}_6) = \{2, 3, 4\}$



$\Gamma(\mathbb{Z}_6)$

Methods

- Our initial approach to this task was to draw as many graphs as we could by hand.
- After many tedious drawings we realized that once you got into larger numbers, the graphs became exceedingly complex to just draw by hand.

- To assist us in understanding the properties of the graphs we created better representations of the graphs.

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- We began to focus on the properties of each number and on it's relation with it's individual graph.

- Even though we made the graphs a little more presentable it still became a little tedious to draw the graphs of $\mathbb{Z}$ mod n for larger n values.

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- Mathematical computer programs became our best friend.

MATHEMATICA!!

Results

- Prime Factorization.
- Categories.

One Prime Factors

- Prime numbers have no Anderson and Livingston Graph.

Two Prime Factors

Represented by p and q

- Distinct Case

Two Prime Factors

Represented by p and q

- Distinct Case

Represented by p^2

- Non-Distinct Case

Distinct Case:

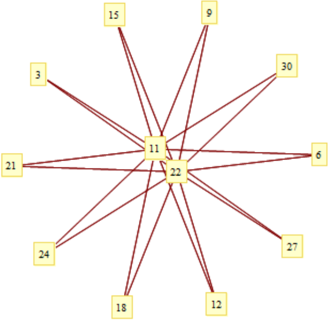

Figure: $\Gamma(\mathbb{Z}_{33})$

Non-Distinct Case:

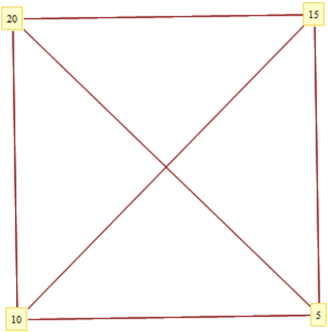

Figure: $\Gamma(\mathbb{Z}_{25})$

Three Prime Factors

This can be represented three different ways.

- p^3
- p^2q
- pqr

First Case : p^3

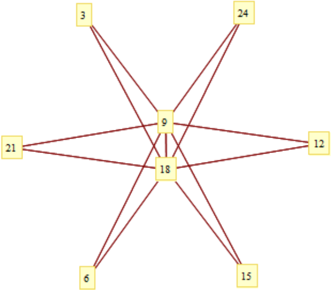

Figure: $\Gamma(\mathbb{Z}_{27})$

Second Case: p^2q

Out[11]=

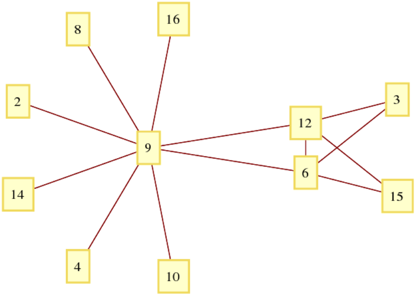

Figure: $\Gamma(\mathbb{Z}_{18})$

Third Case: *pqr*

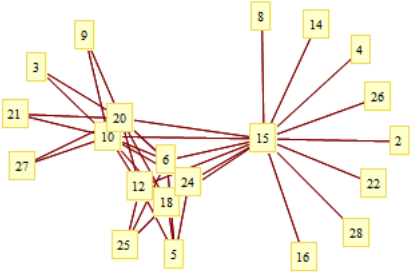

Figure: $\Gamma(\mathbb{Z}_{30})$

Four Prime Factors

This can be represented four different ways.

- p^4
- $p^3 q$
- $p^2 qr$
- $p^2 q^2$

First Case: p^4

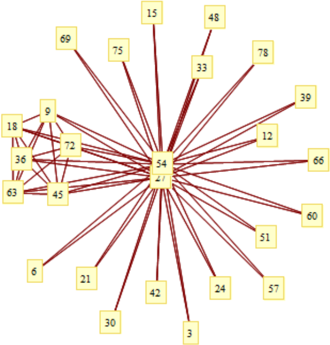

Figure: $\Gamma(\mathbb{Z}_{81})$

Second Case: p^3q

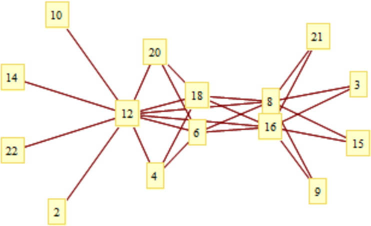

Figure: $\Gamma(\mathbb{Z}_{24})$

Third Case: p^2qr

Out[33]=

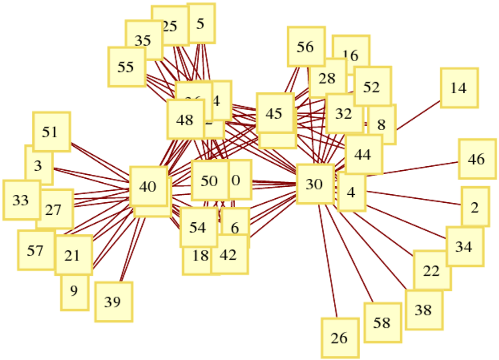

Figure: $\Gamma(\mathbb{Z}_{60})$

Fourth Case: p^2q^2

Out[23]=

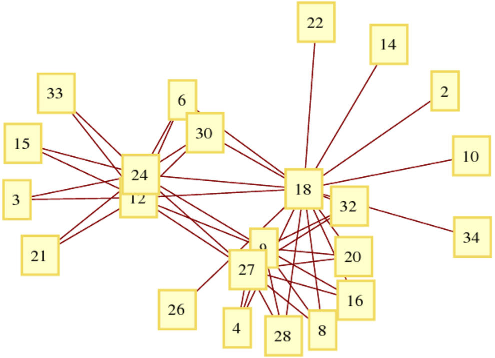

Figure: $\Gamma(\mathbb{Z}_{36})$

Five Prime Factors

Example: p^3q^2

Out[43]=

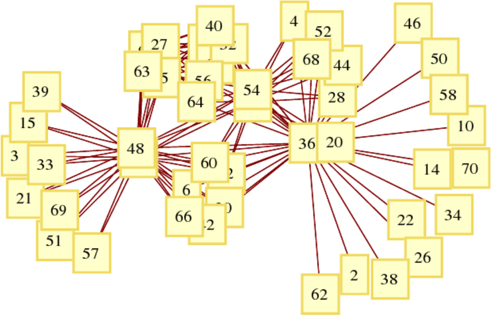

Figure: $\Gamma(\mathbb{Z}_{72})$

Six Prime Factors

Example: p^5q

Out[60]=

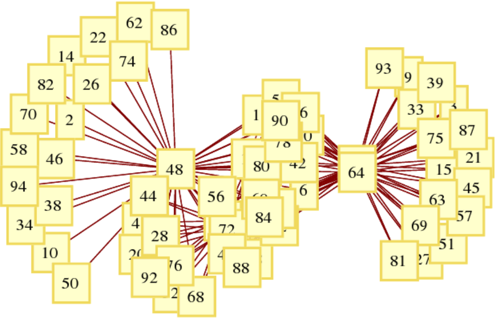

Figure: $\Gamma(\mathbb{Z}_{96})$

Just for some fun....

Example

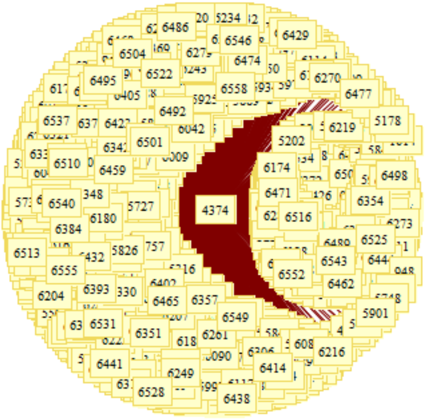

Figure: $\Gamma(\mathbb{Z}_{6561})$

THE CONCLUSION

Based on our study of over 100 zero divisor graphs, we were able to conclude that the complexity of the graph is dependent on the complexity of the prime factorization of the number we are doing modular arithmetic with.

References

1. D. Anderson, P. Livingston, The zero-divisor graph of a commutative ring, J. Algebra, 217 (1999)
2. Dr. Spiroff, Sandra. University of Mississippi.
3. Dr. Chapman, Dave. Louisiana State University.
4. B. Kelly and E. Wilson, Investigating the algebraic structure embedded in zero-divisor graphs, to appear, American J. of Undergraduate Research.