VA

The p-adic valuation of sequences and Wilf's conjecture

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1 Course Description

If p is a prime number, the *p-adic valuation* $\nu_p(n)$ of an integer n is the exponent of the largest power of p that divides n. So for example $\nu_2(24) = 3$, and $\nu_3(24) = 1$.

Studying the p-adic valuation of sequences often provides interesting information. As an example, the p-adic valuation has been used to prove the non-vanishing of specific terms. The reasoning is simple: if we know that $\nu_p(n) = k$ exists as a non-negative integer, then p^{k+1} does not divide n, and so n cannot be 0.

In this course, we will focus on a specific example: the *complementary Bell numbers* $\widetilde{B}(n) = \sum_{k=0}^{n}(-1)^k S(n,k)$. Here $S(n,k)$ are the *Stirling's numbers of the second kind*, defined to be the number of partitions of the set $\{1, 2, \ldots, n\}$ into k non-empty, disjoint sets.

Alternatively, without referring to combinatorics, we can simply define $\widetilde{B}(n)$ to be the n-th derivative at $x = 0$ of the function $f(x) = \exp(1 - e^x)$. A simple check shows that $\widetilde{B}(2) = 0$. H. Wilf conjectured that $\widetilde{B}(n) \neq 0$ for $n > 2$.

We will give a general background on notions in elementary number theory, modular arithmetic, linear algebra, and p-adic valuation, as needed for our study of the Wilf's conjecture. Then we will work on a proof that there is at most one $n > 2$ for which Wilf's conjecture fails. If this integer exists, it will be of the form $24k + 14$. The problem of whether this hypotetical integer exists is currently unsolved. We will present a formulation of the problem in terms of the 2-adic valuation of certain subsequences of $\widetilde{B}(n)$.

This course relies heavily on an experimental mathematics approach to discovery. We will use *Mathematica* to explore the 2-adic valuation patterns of sequences and detect apparent patterns, then provide proofs in the traditional manner.

2 Textbook and technology

1. Notes and handouts provided by the instructor.
2. Suggested, but not required: "Number Theory", by George E. Andrews (from \$3.28 on Amazon), or "Elementary Number Theory", by Efraim P. Armendariz and Stephen J. McAdam (from \$13.05 on half.com).
3. *Mathematica* will be used extensively. Some familiarity with the program will be helpful, but not required, as all necessary code or commands will be provided by the instructor.

3 Undergraduate Projects

1. For $m \geq 1$, let $P_m(r, s) : 0 \leq r, s \leq 2^m - 1$ be the $2^m \times 2^m$ matrix defined by

$$\begin{aligned} P_m(r+1, r) &= 1 \\ P_m(r, r) &= r - 1 \\ P_m(r, r+1) &= -r - 1 \\ P_m(r, s) &= 0 \text{ for } |r - s| > 1. \end{aligned}$$

For example,

$$P_1 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}.$$

and

$$P_2 = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

Use *Mathematica* to study the 2-adic valuation of powers $(P_m)^n$ of the matrices P_m, with special attention to powers of the form $n = 3 \cdot 2^k$. Prove as many of the observed patterns as you can.

2. If k is a non-negative integer, the Pochhammer symbol $x^{(k)}$ is defined by

$$\begin{aligned} x^{(0)} &= 1, \\ x^{(k)} &= x(x+1)(x+2)\cdots(x+k-1), \quad k \geq 1. \end{aligned}$$

Then the set $\{x^{(k)} : 0 \leq k \leq n\}$ is a basis for the vector space of polynomial functions of degree at most n (check this!).

Define monic polynomials $\lambda_n(x)$ (of degree n and with integer coefficients) recursively by

$$\begin{aligned} \lambda_0(x) &= 1, \\ \lambda_{n+1}(x) &= x\lambda_n(x) - \lambda_n(x+1), \end{aligned} \tag{1}$$

and write

$$\lambda_n(x) = \sum_{k=0}^{n} a_n(k) x^{(k)}.$$

Let a_n be the vector $(a_n(k) : k \geq 0)$, where $a_n(k) = 0$ for $k > n$.

(a) Find a matrix R such that $a_{n+1} = Ra_n$.

(b) Let R_m be the $2^m \times 2^m$ matrix consisting of the first 2^m rows and columns of R. Use *Mathematica* to investigate the first column and first row of powers of R_m, comparing the observed values with the complementary Bell numbers. Prove as many experimental observations as you can.

H B

Nonlinear Optimization Methods
of Multivariable Functions
Instructor: Humberto Munoz Barona
Southern University and A& M College

Optimization is a branch of applied mathematics that derives its importance both from the wide variety of its applications and from the availability of efficient algorithms. Mathematically, it refers to the minimization (or maximization) of a given *objective function* of several *decision variables* that satisfy functional *constraints*.

Given a function $\phi(x) : \mathbb{R}^n \to \mathbb{R}$ and a set $S \subset \mathbb{R}$, the problem of finding an $x^* \in \mathbb{R}^n$ that solves

$$\begin{aligned} &\text{minimize } \phi(x) \\ &\text{s.t. } x \in S \\ &f_1(x) \geq 0 \\ &\dots \\ &f_m(x) \geq 0 \end{aligned}$$

is called an optimization problem. ϕ is referred as the *objective function*, S as the *feasible region*, and $f_1, \dots, f_m$ are the functional *constraints*. Problems that lack constraints are called *unconstrained optimization* problems, while others are often called *constrained optimization* problems.

This course will be focused in nonlinear constrained optimization techniques that are commonly used in industrial problems. It will include the following topics: one-dimensional search, the steepest descent method, the conjugate gradient method, the Newton's method, the simplex method, linear programming, nonlinear programming, and least squares. The use of technology is integrated in this course, in particular students will be programing in Mathematica or Matlab.

It is assumed that the student

- has studied two or three semesters of calculus
- has some knowledge using Matlab or Mathematica

Textbooks

Instructor will provide notes and handouts. I recommend that the student purchase one of the following books below, as references.

References

[1] S. Boyd, and L. Vandenberghe. *Convex Optimization*, Cambridge University Press, 2004. It is available in paperback in Amazon for $54.

[2] R. Fletcher. *Practical Methods of Optimization* 2nd. Ed., Wiley, New York, 2001. It is available in paperback in Amazon for $33.85.

[3] P. E. Gill, W. Murray, and M. Wright., *Practical Optimization* by Academic Press, New York, 1981. It is available in paperback in Amazon for $54.

[4] J. Nocedal, and S. Wright. *Numerical Optimization*, Springer, New York, 2000. It is available in hardcover in Amazon for $46.97.

Project 1: Data analysis

Students will use a Matlab program that performs the Levenberg-Marquardt method. They will input the data and any other quantities that they may need, and the model as an M-file. A heuristic method for changing the regularization parameter must be used.

Project 2: Maximizing profit and minimizing the cost of a product

This is a multi-objective optimization problem that is very common in several fields (product and process design, finance, the oil and gas industry, etc). Students will engage in the solution of this problem by constructing a single aggregate objective function (AOF) and using Mathematica.

MD

An Introduction to Frames

instructor: Mark G. Davidson

Suppose you are interested in sending a signal over a communication system. We think of that signal as a vector in a vector space. The way it gets transmitted is as a sequence of coefficients that represent the signal in terms of some basis. If the vector space has an inner product and the basis is an orthonormal basis then the coefficients that you transmit may be easily and quickly computed and there is no significant time delay. However, being an orthonormal basis is very restrictive. What if one of the coefficients gets lost in the transmission. That piece of information cannot be reconstructed. So we would like our system to have some redundancy: if one piece gets lost, that information can be pieced together from what does get through. That's where a frame comes in. A frame is a spanning set that is not necessarily linearly independent. A Parseval frame retains the very convenient property that coefficients for expanding vectors can be computed using the inner product. Frames are being utilized in industries including celluar phones, medical imaging, and recognition and identification software.

This course is a systematic study of frames (mostly in finite dimensional vector spaces). A prerequisite is a sophomore level course in linear algebra, such as is taught in Math 2065 at LSU. However, we will review and even redo many important standard facts. Keep in mind that this will be a math course and not an engineering course on signal processing.

Projects

Both projects are somewhat exploratory in nature. The two topics listed below are for you to read about, explore, and discover. I am not going to list at this time any specific goals, unknown problems, or major theorems that you must accomplish.

Project MD 1: Harmonic and Group Frames Frames with special structures are important in many applications. Harmonic and Group Frames are obtained by applying a unitary group representation to some fixed vector in the inner product space. When the resulting set of vectors form a Frame it is called a Group Frame. In a similar manner Harmonic frames are generated from a single starting point. This project will explore the properties of Harmonic and Group Frames.

Project MD 2: Dual and Orthogonal Frames Given a frame that is not necessarily a Parseval frame (don't worry you will learn the meaning of this jargon) one can construct the so-called dual frame and orthogonal frame. This project will explore their properties.

Text and References

It will be very helpful for you to purchase the text "Frames for Undergraduates" by Deguang Han, Keri Kornelson, David Larson, and Eric Weber. I will be mainly lecturing out of this text. I purchased the book on Amazon.com for $45.00. You might check if your library has a copy that you can check out for the program.

Another reference that is handy is "An introduction to Frames and Riesz Bases" by Ole Christensen.

LS

Conic Sections & Geometric Constructions

Larry Smolinsky, Professor at LSU.
VIGRE summer SMILE course.

The Ancient Greeks introduced the systematic study of mathematics as its own field and introduced rigorous argument. Perhaps most notable are the thirteen books of Euclid's Elements, which proceed by the method of construction. The Ancient Greeks had a hierarchy of geometric constructions. Those called planar constructions were made with a straightedge and compass. These constructions embody Euclid's axioms for plane geometry. Those called solid constructions allow the use of conic sections.

We will discuss geometric constructions and pay particular attention to the classical problems:

Trisecting an angle
Constructing a cube twice the volume of a given cube
Construct a square equal in area to the area of a given circle

We will examine geometric constructions from the modern viewpoint via algebra as well as geometry. Resolving these ancient questions and related questions is among the triumphs of modern abstract mathematics. Along the way, we will discuss classical geometry, complex numbers, polynomial equations, fields, and conic sections.

Projects. The first project is for high school students.

1. **Complex numbers.** In working on this project, students will learn about the complex numbers and their geometric representation. In examining the solutions to $x^n - 1 = 0$, you will find a surprising formula concerning the regular polygons.

Start by defining the complex numbers (denoted **C**) as the set of a + bi where a,b are real numbers.
Define addition and multiplication, complex conjugate, and modulus.
Show that addition is commutative and associative:
$a,b,c \in \mathbf{C} \Rightarrow a+b=b+a$ and $a+(b+c)=(a+b)+c$.
Show that multiplication is commutative and associative:
$a,b,c \in \mathbf{C} \Rightarrow ab=ba$ and $a(bc)=(ab)c$.
Show that multiplication and addition are distributive:
$a,b,c \in \mathbf{C} \Rightarrow a(b+c)=ab+ac$.
Show that there is an additive identity (zero) and a multiplicative identity (one).
Show that every complex number ($a \in \mathbf{C}$) has an additive inverse ($-a \in \mathbf{C}$) and that every nonzero number ($a \in \mathbf{C}$) has a multiplicative inverse ($a^{-1} \in \mathbf{C}$). How do you divide one complex number by another?

Prove that for all $a,b \in \mathbf{C}$, $ab=0 \Rightarrow a = 0$ or $b = 0$.
If p(x) is a polynomial, then how many solutions may the equation p(x)=0 have.
What are the solutions to $x^2-2=0$, $x^2+5x-4=0$, and $x^4-1=0$?

Represent the complex numbers on the plane like you represent real numbers on the line.
Give a geometric interpretation on the plane of adding complex numbers.
Represent a complex element in the following polar form $a = m(\cos\theta + i\sin\theta)$.
Give a geometric interpretation on the plane of multiplying complex numbers.
Give a geometric interpretation on the plane of taking square roots of complex numbers.
Find the solutions to $x^3-1=0$ and $x^8-1=0$. Plot them on the plane. Factor the polynomials $x^3-1=0$ and $x^8-1=0$.
Find the solutions to $x^9-1=0$, plot them on the plane, and factor the polynomial.
Suppose that $a_0, a_1, a_2, \ldots, a_8$ are the vertices of a regular nine-gon. Compute the product of the distances from a_0 to the other vertices, i.e., the distance from a_0 to a_1 times the distance from a_0 to a_2 times the distance from a_0 to a_3 etc.
Do the same for the regular n-gon.

2. **Sunrise on Mercury**. Proposed by Andrew McDaniel. Make a Mathematica program that shows the path of the sun during a day on Mercury. You should initially make some simplifying assumptions, e.g., the axis of rotation of mercury is perpendicular to the plane of the orbit, the orbit of mercury is perfectly elliptical, the observer is standing on the equator, and any other idealizing assumptions that may be helpful. You should write the program with the initial position of the planet in the orbit as a variable.

3. **Curves and sectioning angles**. Let A=(0,0) and B=(1,0). Define a curve Γ to consist of those points C=(x,y) in the first and second quadrants such that $\angle ABC = 2\ \angle BAC$. Show that Γ is one branch of a hyperbola (that is missing one point). Find the equation of the hyperbola. Try to find geometric interpretations of the asymptotes, the focus, and directrix. Show that this one hyperbola along with the use of a straight edge and compass is enough to trisect any acute angle. You may show it by adapting an argument by Pappus given in Sir Thomas Little Heath's A History of Greek Mathematics, volume 1, pp. 241-243.

Define other curves Γ_n as the points (x,y) in the first and second quadrants such that $\angle ABC = n\ \angle BAC$ where A=(0,0), B=(1,0), and C=(x,y). What information can you discover about these curves? Show how to section an angle into n+1 equal parts using Γ_n along with a straight edge and compass.