Sampling Theory

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- has applications in image reconstruction and cd storage.

We will use $V = \mathbb{P}_N(\mathbb{R})$, the set of polynomials of degree N or less.

Background Information

Lemma (The Alpha Lemma)

Let V be a finite dimensional vector space of dimension n. If $\{v_i\}_{i=1}^{n}$ is a set vectors that span V and, for all i, $v_i \in V$, then $\{v_i\}_{i=1}^{n}$ is a basis.

Necessary Info

Definition

If we let $\{v_1 \ldots v_k\}$ be any set of vectors, it is then classified as a **frame** if there are numbers $A, B > 0$ such that $\forall\, v \,\epsilon V$ the following inequality stands true

$$A\|v\|^2 \leq \sum_{i=1}^{k} \mid (v, v_i) \mid^2 \leq B\|v\|^2.$$

Necessary Info

For simplistic purposes, the following lemma will be what is used to define a frame.

Lemma

The set $\{v_1, \ldots, v_k\}$ is said to be a frame if and only if it is a spanning set of V.

Necessary Info

For every frame there exist what is known as a **dual frame**.

Theorem (The Dual Theorem)

Suppose $\{v_i\}_{i=1}^{k}$ is a frame, then the dual frame $\{w_i\}_{i=1}^{k}$ of V there exist for all $v \in V$

$$v = \sum_{i=1}^{k} (v|v_i) w_i = \sum_{i=1}^{k} (v|w_i) v_i .$$

Necessary Info

It is necessary to introduce what is known as a **analysis operator**.

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Definition

The analysis operator, denoted as Θ, is a linear map from the vector space V to $\mathbb{R}^k$ such that for a given $v \in V$ and frame $\{v_i\}_{i=1}^k$

$$\Theta(v) = \begin{pmatrix} (v|v_1) \\ (v|v_2) \\ \vdots \\ (v|v_k) \end{pmatrix}$$

Necessary Info

The frame operator, denoted S, has the following properties

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The following proposition and theorem, along with the properties of S^{-1}, allow for S^{-1} to be computed explicitly over a set $\mathbb{X}$.

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The frame operator, denoted S, has the following properties

- $S = \Theta^*\Theta$, where Θ^* is the adjoint of Θ.
- S is invertible and is equal to its own adjoint and thus is self-adjoint,
- S^{-1} is self-adjoint.

The following proposition and theorem, along with the properties of S^{-1}, allow for S^{-1} to be computed explicitly over a set $\mathbb{X}$.

Definition

We say the set $\mathbb{X}$ can be a set of uniqueness for $\mathbb{P}_N$ if for every $f, g \in \mathbb{P}_N$, $f|_{\mathbb{X}} = g|_{\mathbb{X}} \Rightarrow f = g$.

Necessary Info

With every dual frame there also exist what is known as the **canonical dual frame**, where $w_i = S^{-1} v_i$.

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Proposition

If we define $\mathbb{P}_N := \{f : \mathbb{R} \to \mathbb{R} : f(x) = \sum_{n=0}^{N} a_n x^n, a_n \epsilon \mathbb{R}\}$*, which is a set of polynomials of degree* N *or less, then set* $\mathbb{X}$ *can be a set of uniqueness for* $\mathbb{P}_N$ *if and only if* $\Theta_{\mathbb{X}}$ *is injective.*

As a consequence,

As a consequence,

Theorem

Let $\mathbb{X} = \{x_0, x_1, \ldots, x_N\} \subset \mathbb{R}$; *then* $\mathbb{X}$ *is a set of uniqueness for* $\mathbb{P}_N$.

Necessary Info

By the definiton of S, the matrix representation of S can be derived using the matrix previously defined. Thus

$$[S] = \begin{bmatrix} N+1 & \sum_{i=0}^{N} x_i & \sum_{i=0}^{N} x_i^2 & \cdots & \sum_{i=0}^{N} x_i^N \\ \sum_{i=0}^{N} x_i & \sum_{i=0}^{N} x_i^2 & \sum_{i=0}^{N} x_i^3 & \cdots & \sum_{i=0}^{N} x_i^{N+1} \\ \sum_{i=0}^{N} x_i^2 & \sum_{i=0}^{N} x_i^3 & \sum_{i=0}^{N} x_i^4 & \cdots & \sum_{i=0}^{N} x_i^{N+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sum_{i=0}^{N} x_i^N & \sum_{i=0}^{N} x_i^{N+1} & \sum_{i=0}^{N} x_i^{N+2} & \cdots & \sum_{i=0}^{N} x_i^{2N} \end{bmatrix}$$

Necessary Info

Example: Consider $\mathbb{P}_2$ and $\mathbb{X} = \{-2,\ 0,\ 1\}$. Let our frame be denoted as $\{(3,0,0),\ (0,1,0),\ (0,0,2)\}$ where for a polynomial $(a,\ b,\ c)$ corresponds to $a + bx + cx^2$. The following Mathematica commands provide an outline.

Necessary Info

$r_1 = -2;$
$r_2 = 0;$
$r_3 = 1;$

theta $= \left\{ \{1, r_1, r_1^2\}, \{1, r_2, r_2^2\}, \{1, r_3, r_3^2\} \right\};$

%16//MatrixForm

$$\begin{pmatrix} 1 & -2 & 4 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

adjoint $=$Transpose$\left[\left\{ \{1, r_1, r_1^2\}, \{1, r_2, r_2^2\}, \{1, r_3, r_3^2\} \right\} \right];$

%17//MatrixForm

$$\begin{pmatrix} 1 & 1 & 1 \\ -2 & 0 & 1 \\ 4 & 0 & 1 \end{pmatrix}$$

Necessary Info

S =adjoint .theta;

%22//MatrixForm

$$\begin{pmatrix} 3 & -1 & 5 \\ -1 & 5 & -7 \\ 5 & -7 & 17 \end{pmatrix}$$

InS=Inverse[S];

%29//MatrixForm

$$\begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{13}{18} & \frac{4}{9} \\ -\frac{1}{2} & \frac{4}{9} & \frac{7}{18} \end{pmatrix}$$

Necessary Info

InS.{3,0,0}

$\left\{3, -\frac{3}{2}, -\frac{3}{2}\right\}$

InS.{0,2,0}

$\left\{-1, \frac{13}{9}, \frac{8}{9}\right\}$

InS.{0,0,1}

$\left\{-\frac{1}{2}, \frac{4}{9}, \frac{7}{18}\right\}$

Necessary Info

Necessary Info

As continued,

Necessary Info

As continued,

Definition

We call a set of uniqueness a **set of sampling** if there exists a "reasonable" algorithm for computing a function f from $f|_{\mathbb{X}}$

Theorem

Let $\mathbb{X} = \{x_0, x_1, \ldots, x_N\} \subset \mathbb{R}$; *then* $\mathbb{X}$ *is a set of sampling for* $\mathbb{P}_N$.

Necessary Info

Proof

Assuming for some $f \in \mathbb{P}_N$ for which we know the values of f at points in $\mathbb{X}$, it suffices to must provide a reconstruction algorithm to find f.

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Assuming for some $f \in \mathbb{P}_N$ for which we know the values of f at points in $\mathbb{X}$, it suffices to must provide a reconstruction algorithm to find f.

Necessary Info

Proof cont.

Necessary Info

Proof cont.

One such algorithm is known as Lagrange interpolation, which generates the following polynomials

Necessary Info

Proof cont.

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$$p_{x_j}(x) = \prod_{k=0,\, k\neq j}^{N} \frac{(x - x_k)}{(x_j - x_k)}.$$

Necessary Info

Proof cont.

Necessary Info

Proof cont.

Each polynomial has degree N and satisfies the conditions

Necessary Info

Proof cont.

Each polynomial has degree N and satisfies the conditions

$$p_{x_j}(x_k) = \delta_{jk} = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases}.$$

Necessary Info

Proof cont.

Thus

Necessary Info

Proof cont.

Thus

- $g(x) = \sum_{j=0}^{N} f(x_j) p_{x_j}(x) \in \mathbb{P}_N$.

Necessary Info

Proof cont.

Thus

- $g(x) = \sum_{j=0}^{N} f(x_j) p_{x_j}(x) \in \mathbb{P}_N$.
- $f(x_j) = g(x_j)$ for all j.

Necessary Info

Proof cont.

Thus

- $g(x) = \sum_{j=0}^{N} f(x_j) p_{x_j}(x) \in \mathbb{P}_N$.
- $f(x_j) = g(x_j)$ for all j.
- By previous theorem $f(x) = g(x)$.

Necessary Info

Proof cont.

Thus

- $g(x) = \sum_{j=0}^{N} f(x_j)p_{x_j}(x) \in \mathbb{P}_N$.
- $f(x_j) = g(x_j)$ for all j.
- By previous theorem $f(x) = g(x)$.
- $f(x) = \sum_{j=0}^{N} f(x_j)p_{x_j}(x)$.

$\square$

Necessary Info

Proof cont.

Thus

- $g(x) = \sum_{j=0}^{N} f(x_j)p_{x_j}(x) \in \mathbb{P}_N$.
- $f(x_j) = g(x_j)$ for all j.
- By previous theorem $f(x) = g(x)$.
- $f(x) = \sum_{j=0}^{N} f(x_j)p_{x_j}(x)$.

$\square$

From this point forward, the Lagrange polynomials will be denoted as $\{p_j(x)\}_{j=0}^{N}$ where $p_j = p_{x_j}$.

Necessary Info

Theorem (Riesz Representation Theorem)

If φ is a linear functional on a finite-dimensional inner product space, $\mathcal{V}$ (meaning, $\varphi : \mathcal{V} \overset{linear}{\rightarrow} \mathbb{F}$), then there exist $v_\varphi \in \mathcal{V}$ such that

$$\varphi(v) = (v | v_\varphi)$$

Necessary Info

Claim

The Lagrange polynomials $\{p_j(x)\}_{j=0}^{N}$ *form a basis.*

Proof

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- Suffices to show $\{p_j(x)\}_{j=0}^{N}$ spans $\mathbb{P}_N$.

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Proof

- Suffices to show $\{p_j(x)\}_{j=0}^{N}$ spans $\mathbb{P}_N$.
- Consider an arbitrary polynomial $p(x)$ in $\mathbb{P}_N$. Then $p(x) = \sum_{j=0}^{N} c_j\, x^j$.

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The Lagrange polynomials $\{p_j(x)\}_{j=0}^N$ *form a basis.*

Proof

- Suffices to show $\{p_j(x)\}_{j=0}^N$ spans $\mathbb{P}_N$.
- Consider an arbitrary polynomial $p(x)$ in $\mathbb{P}_N$. Then $p(x) = \sum_{j=0}^N c_j x^j$.
- Allow $c_j = d_j \sum_{k=0}^N p_{jk}$,

Necessary Info

cont.

$$p(x) \quad =$$

Necessary Info

cont.

$$p(x) = \sum_{j=0}^{N} x^j \, d_j \sum_{k=0}^{N} p_{jk}$$

Necessary Info

cont.

$$\begin{aligned} p(x) &= \sum_{j=0}^{N} x^j\, d_j \sum_{k=0}^{N} p_{jk} \\ &= \sum_{j=0}^{N} d_j \sum_{k=0}^{N} p_{jk}\, x^j \end{aligned}$$

Necessary Info

cont.

$$\begin{aligned} p(x) &= \textstyle\sum_{j=0}^{N} x^j\, d_j \sum_{k=0}^{N} p_{jk} \\ &= \textstyle\sum_{j=0}^{N} d_j \sum_{k=0}^{N} p_{jk}\, x^j \\ &= \textstyle\sum_{j=0}^{N} d_j \sum_{k=0}^{N} p_{jk}\, x^k \end{aligned}$$

Necessary Info

cont.

$$\begin{aligned} p(x) &= \textstyle\sum_{j=0}^{N} x^j \, d_j \sum_{k=0}^{N} p_{jk} \\ &= \textstyle\sum_{j=0}^{N} d_j \sum_{k=0}^{N} p_{jk} \, x^j \\ &= \textstyle\sum_{j=0}^{N} d_j \sum_{k=0}^{N} p_{jk} \, x^k \\ &= \textstyle\sum_{j=0}^{N} d_j \, p_j(x) \end{aligned}$$

So $p(x)$ can be written as a linear combination of Lagrange polynomials, so $\{p_j(x)\}_{j=0}^{N}$ spans $\mathbb{P}_N$ and, by the Alpha Lemma, is a basis. □

Necessary Info

Necessary Info

- The unique canonical dual frame can be used to express any set frame, using S^{-1}.

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- The unique canonical dual frame can be used to express any set frame, using S^{-1}.
- We can further use the *Riesz Representation Theorem* to connect reconstructive vectors used in sampling.

The Discovery

Assuming we use the standard dot product as our inner product, we shall explore the certain properties. As previously mentioned, $\Theta_{\mathbb{X}}(f)$ evaluates the function f at each of the points in $\mathbb{X}$. Thus by the Riesz Representation Theorem,

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Assuming we use the standard dot product as our inner product, we shall explore the certain properties. As previously mentioned, $\Theta_{\mathbb{X}}(f)$ evaluates the function f at each of the points in $\mathbb{X}$. Thus by the Riesz Representation Theorem,

$$\begin{pmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_N) \end{pmatrix} = \begin{pmatrix} (f|\, f_{x_0}) \\ (f|\, f_{x_1}) \\ \vdots \\ (f|\, f_{x_N}) \end{pmatrix}.$$

The Discovery

$f(x) = \sum_{j=0}^{N} a_j x^j$, $f_{x_j}(x) = \sum_{j=0}^{N} b_j x^j$.

$$f(x_i) = a_0 + a_1 x_i + a_2 x_i^2 + \ldots + a_N x_i^N$$

$$\begin{aligned} (f \mid f_{x_i}) &= \sum_{j=0}^{N} a_j\, b_j \\ &= a_0\, b_0 + a_1 b_1 + a_2 b_2 + \ldots + a_N b_N \end{aligned}$$

Solving for the b_j's, we get that $b_j = x_i^j$

The Discovery

Claim

$\{f_{x_i}\}_{i=0}^{N}$ *is a frame*

Proof

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- Suffices to show that the set spans $\mathbb{P}_N$.

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Proof

- Suffices to show that the set spans $\mathbb{P}_N$.
- Let $p(x) = \sum_{j=0}^{N} a_j x^j$, where we take $\{x^j\}_{j=0}^{N}$ to be the standard basis for $\mathbb{P}_N$ and $a_j \in \mathbb{R}$.

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- Suffices to show that the set spans $\mathbb{P}_N$.
- Let $p(x) = \sum_{j=0}^{N} a_j x^j$, where we take $\{x^j\}_{j=0}^{N}$ to be the standard basis for $\mathbb{P}_N$ and $a_j \in \mathbb{R}$.
- Allowing $a_j = c_j \sum_{r=0}^{N} x_j^r$

The Discovery

cont.

$$p(x) \quad =$$

The Discovery

cont.

$$
\begin{aligned}
p(x) &= \sum_{j=0}^{N} x^j c_j \sum_{r=0}^{N} x_j^r \\
&=
\end{aligned}
$$

The Discovery

cont.

$$
\begin{aligned}
p(x) &= \textstyle\sum_{j=0}^{N} x^j c_j \sum_{r=0}^{N} x_j^r \\
&= \textstyle\sum_{j=0}^{N} c_j \sum_{r=0}^{N} x_j^r \, x^j \\
&=
\end{aligned}
$$

The Discovery

cont.

$$\begin{aligned} p(x) &= \textstyle\sum_{j=0}^{N} x^j c_j \sum_{r=0}^{N} x_j^r \\ &= \textstyle\sum_{j=0}^{N} c_j \sum_{r=0}^{N} x_j^r \, x^j \\ &= \textstyle\sum_{j=0}^{N} c_j \sum_{r=0}^{N} x_j^r \, x^r \\ &= \end{aligned}$$

The Discovery

cont.

$$
\begin{aligned}
p(x) &= \sum_{j=0}^{N} x^j c_j \sum_{r=0}^{N} x_j^r \\
&= \sum_{j=0}^{N} c_j \sum_{r=0}^{N} x_j^r \, x^j \\
&= \sum_{j=0}^{N} c_j \sum_{r=0}^{N} x_j^r \, x^r \\
&= \sum_{j=0}^{N} c_j \, f_{x_j}(x)
\end{aligned}
$$

Thus we can write any polynomial in $\mathbb{P}_N$ as a linear combination of of vectors in $\{f_{x_i}\}_{i=0}^{N}$, so $\{f_{x_i}\}_{i=0}^{N}$ spans $\mathbb{P}_N$ and is a frame. $\square$

The Discovery

Since the Lagrange polynomials were shown to be a basis, the Dual Theorem can be used to confirm if $\{f_{x_i}\}_{i=0}^{N}$ is a dual frame and thus the canonical dual frame by the following proposition.

The Discovery

Since the Lagrange polynomials were shown to be a basis, the Dual Theorem can be used to confirm if $\{f_{x_i}\}_{i=0}^{N}$ is a dual frame and thus the canonical dual frame by the following proposition.

Proposition (The Indecent Proposition)

Let $\{x_i\}_{i=0}^{N}$ be a basis for a finite-dimensional inner product space. Then its dual frame is unique.

The Discovery

$f(x) = \sum_{j=0}^{N} f(x_j)\, p_j(x)$, $p_j(x) = j^{th}$ Lagrange polynomial written in the standard basis of $\mathbb{P}_N$ defined as $p_j(x) = \sum_{k=0}^{N} p_{jk}\, x^k$. By the Riesz Representation Theorem, $f(x) = \sum_{j=0}^{N} (f|\, f_{x_j})\, p_j(x)$.

The Discovery

$$(f|\, f_{x_j}) \quad =$$

The Discovery

$$
\begin{aligned}
(f \,|\, f_{x_j}) &= \sum_{r=0}^{N} a_r\, b_r \\
&=
\end{aligned}
$$

The Discovery

$$
\begin{aligned}
(f \,|\, f_{x_j}) &= \sum_{r=0}^{N} a_r \, b_r \\
&= \sum_{r=0}^{N} (f(x_r) \sum_{k=0}^{N} x_r^k)(x_j^r) \\
&=
\end{aligned}
$$

The Discovery

$$
\begin{aligned}
(f \,|\, f_{x_j}) &= \sum_{r=0}^{N} a_r\, b_r \\
&= \sum_{r=0}^{N} (f(x_r) \sum_{k=0}^{N} x_r^k)(x_j^r) \\
&= \sum_{r=0}^{N} \sum_{k=0}^{N} f(x_r)\, x_j^r\, x_r^k
\end{aligned}
$$

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$$(f \,|\, p_j) \quad =$$

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$$
\begin{aligned}
(f \,|\, p_j) &= \sum_{r=0}^{N} a_r \, p_{jr} \\
&=
\end{aligned}
$$

The Discovery

$$
\begin{aligned}
(f \mid p_j) &= \sum_{r=0}^{N} a_r \, p_{jr} \\
&= \sum_{r=0}^{N} (f(x_r) \sum_{k=0}^{N} x_r^k)(p_{jr}) \\
&=
\end{aligned}
$$

The Discovery

$$
\begin{aligned}
(f \mid p_j) &= \sum_{r=0}^{N} a_r \, p_{jr} \\
&= \sum_{r=0}^{N} (f(x_r) \sum_{k=0}^{N} x_r^k)(p_{jr}) \\
&= \sum_{r=0}^{N} \sum_{k=0}^{N} f(x_r) \, x_r^k \, p_{jr}
\end{aligned}
$$

The Discovery

Thus

$$\sum_{j=0}^{N} (f \,|\, f_{x_j})\, p_j(x) \quad =$$

The Discovery

Thus

$$\sum_{j=0}^{N} (f \,|\, f_{x_j})\, p_j(x) \quad = \quad \sum_{j=0}^{N} (\sum_{r=0}^{N} \sum_{k=0}^{N} f(x_r)\, x_j^r\, x_r^k) p_j(x)$$

The Discovery

Thus

$$\begin{aligned}\sum_{j=0}^{N}(f|\, f_{x_j})\, p_j(x) &= \sum_{j=0}^{N}(\sum_{r=0}^{N}\sum_{k=0}^{N} f(x_r)\, x_j^r\, x_r^k)p_j(x) \\ &= \sum_{j=0}^{N}(\sum_{r=0}^{N}\sum_{k=0}^{N} f(x_r)\, x_j^r\, x_r^k)(\sum_{s=0}^{N} p_{js}\, x^s)\end{aligned}$$

The Discovery

Thus

$$\begin{aligned}
\sum_{j=0}^{N} (f \,|\, f_{x_j})\, p_j(x) &= \sum_{j=0}^{N} \Big(\sum_{r=0}^{N} \sum_{k=0}^{N} f(x_r)\, x_j^r\, x_r^k\Big) p_j(x) \\
&= \sum_{j=0}^{N} \Big(\sum_{r=0}^{N} \sum_{k=0}^{N} f(x_r)\, x_j^r\, x_r^k\Big)\Big(\sum_{s=0}^{N} p_{js}\, x^s\Big) \\
&= \sum_{j=0}^{N} \sum_{r=0}^{N} \sum_{k=0}^{N} \sum_{s=0}^{N} f(x_r)\, x_j^r\, x_r^k\, p_{js}\, x^s
\end{aligned}$$

The Discovery

Whereas

$$\sum_{j=0}^{N} (f \,|\, p_j)\, f_{x_j}(x) \quad =$$

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$$\sum_{j=0}^{N} (f \,|\, p_j)\, f_{x_j}(x) \quad = \quad \sum_{j=0}^{N} (\sum_{r=0}^{N} \sum_{k=0}^{N} f(x_r)\, x_r^k\, p_{jr})\, f_{x_j}(x)$$

The Discovery

Whereas

$$\begin{aligned}\sum_{j=0}^{N} (f|\, p_j)\, f_{x_j}(x) &= \sum_{j=0}^{N} (\sum_{r=0}^{N} \sum_{k=0}^{N} f(x_r)\, x_r^k\, p_{jr})\, f_{x_j}(x) \\ &= \sum_{j=0}^{N} (\sum_{r=0}^{N} \sum_{k=0}^{N} f(x_r)\, x_r^k\, p_{jr})(\sum_{s=0}^{N} x_j^s x^s)\end{aligned}$$

The Discovery

Whereas

$$\begin{aligned}
\sum_{j=0}^{N} (f \,|\, p_j)\, f_{x_j}(x) &= \sum_{j=0}^{N} \Big(\sum_{r=0}^{N} \sum_{k=0}^{N} f(x_r)\, x_r^k\, p_{jr}\Big)\, f_{x_j}(x) \\
&= \sum_{j=0}^{N} \Big(\sum_{r=0}^{N} \sum_{k=0}^{N} f(x_r)\, x_r^k\, p_{jr}\Big)\Big(\sum_{s=0}^{N} x_j^s x^s\Big) \\
&= \sum_{j=0}^{N} \sum_{r=0}^{N} \sum_{k=0}^{N} \sum_{s=0}^{N} f(x_r)\, x_j^s\, x_r^k\, p_{js}\, x^s
\end{aligned}$$

The Discovery

Whereas

$$
\begin{aligned}
\sum_{j=0}^{N} (f|\, p_j)\, f_{x_j}(x) &= \sum_{j=0}^{N} (\sum_{r=0}^{N} \sum_{k=0}^{N} f(x_r)\, x_r^k\, p_{jr})\, f_{x_j}(x) \\
&= \sum_{j=0}^{N} (\sum_{r=0}^{N} \sum_{k=0}^{N} f(x_r)\, x_r^k\, p_{jr})(\sum_{s=0}^{N} x_j^s x^s) \\
&= \sum_{j=0}^{N} \sum_{r=0}^{N} \sum_{k=0}^{N} \sum_{s=0}^{N} f(x_r)\, x_j^s\, x_r^k\, p_{js}\, x^s \\
&= \sum_{j=0}^{N} \sum_{r=0}^{N} \sum_{k=0}^{N} \sum_{s=0}^{N} f(x_r)\, x_j^r\, x_r^k\, p_{js}\, x^s
\end{aligned}
$$

The Discovery

Since $f(x) = \sum_{j=0}^{N} (f \mid f_{x_j})\, p_j(x) = \sum_{j=0}^{N} (f \mid p_j)\, f_{x_j}(x)$ for all f, $\{f_{x_i}\}_{i=0}^{N}$ is not only a dual frame for the Lagrange polynomials, by the Indecent proposition, it is the only dual frame and therefore is it's canonical dual frame.

In Conclusion

Sampling theory is an interesting topic that shows the variability of linear algebra in its ways of approach. We've shown how approaching sampling from two different perspectives, that of linear transforms and inner products, can yield results that appear totally different but are actually the same.

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THE END!!!!!!!!!!!!!!!!!!!