PARSEVAL FRAME CONSTRUCTION

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Abstract. General methods of constructing Parseval frames are desirable in many kinds of signal processing applications, as well as for use by mathematical researchers in frame theory. We begin by examining how the orthonormality of the row space of certain matrices implies the column space is a Parseval frame. We then use these results to study harmonic frames, which are frames arising from generalizations of the discrete Fourier transform. We then discuss a process, analogous to the well-known Gram-Schmidt orthogonalization process, which converts a frame into a Parseval frame. All our work takes place in finite dimensional spaces.

1. Background Information

We begin with a small review of linear algebra and a small introduction to frames.

Definition 1.1. We say a complex (real) finite dimensional vector space is a **Hilbert space**, V, if it is equipped with an inner product $\langle f|g\rangle$, that is, a map of $V \times V \to \mathbb{C}$ which satisfies

$$\langle f+g|h\rangle = \langle f|h\rangle + \langle g|h\rangle \tag{1.1}$$

$$\langle \alpha f|g\rangle = \alpha\langle f|g\rangle \tag{1.2}$$

$$\langle h|g\rangle = \overline{\langle g|h\rangle} \tag{1.3}$$

$$\langle f|f\rangle = 0 \text{ then } f = 0 \tag{1.4}$$

for all scalars α and f, g, h in V.

Through this article, all Hilbert spaces will be finite dimensional.

Definition 1.2. Recall that operators in a finite dimensional Hilbert space have adjoints, by the Riesz Representation Theorem [1], which we denote T^*. We say T is a normal operator on V if $TT^* = T^*T$. That is, T commutes with its adjoint.

Definition 1.3. An operator T on V is self-adjoint if $T = T^*$.

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Definition 1.4. If T is an operator on V, T can be represented by a matrix, denoted $[T]$ or $[a_{ij}]$.

Definition 1.5. A positive operator T is a self-adjoint operator on V such that $\langle Tv|v\rangle \geq 0$ for any $v \in V$.

Theorem 1.6 (Spectral Theorem)**.** [1] *Let T be a normal linear operator on a finite-dimensional Hilbert space V with distinct eigenvalues λ_1, λ_2, ... , λ_k, and let P_j be the orthogonal projection of V onto the eigenspace $E_{\lambda j}$ for $1 \leq j \leq k$. Then the following are true:*

$$V = E_{\lambda 1} \oplus E_{\lambda 2} \oplus \ldots \oplus E_{\lambda k} \tag{1.5}$$

$$P_i P_j = 0 \textit{ for } i \neq j \tag{1.6}$$

$$P_1 + P_2 + \ldots + P_k = I \tag{1.7}$$

$$T = \lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_k P_k \tag{1.8}$$

Definition 1.7. [1] A **frame** for a Hilbert space V is a sequence of vectors $\{x_i\} \subset V$ for which there exist constants $0 < A \leq B < \infty$ such that, for every $x \in V$,

$$A||x||^2 \leq \sum_i |\langle x|x_i\rangle|^2 \leq B||x||^2.$$

A frame is a **Parseval frame** if $A = B = 1$.

Proposition 1.8. [1] *A collection of vectors $\{x_i\}_{i=1}^k$ is a Parseval frame for a Hilbert space V if and only if the following formula holds for every x in V:*

$$x = \sum_{i=1}^{k} \langle x|x_i\rangle x_i.$$

*This equation is called the **reconstruction formula** for a Parseval frame.*

Theorem 1.9. [1] *Suppose that V is a finite-dimensional Hilbert space and $\{x_i\}_{i=0}^k$ is a finite collection of vectors from V. Then the following statements are equivalent:*

- *$\{x_i\}_{i=0}^k$ is a frame for V*
- *span $\{x_i\}_{i=0}^k = V$.*

Definition 1.10. [2] Let $\{a_i\}_{i=1}^k$ be a set for V. Let $\Theta_a : V \to \mathbb{C}^k$ be defined as $\Theta_a(v) = \begin{pmatrix} \langle v|a_1\rangle \\ \vdots \\ \langle v|a_k\rangle \end{pmatrix}$ for $v \in V$. We call Θ_f the **analysis operator** for $\{a_i\}$.

Definition 1.11. [1] Let $\{a_i\}_{i=1}^k$ be a set for V. Let $\Theta_a^* : \mathbb{C}^k \to V$ be the adjoint of Θ_a. We call Θ_a^* the **reconstruction operator** for $\{a_i\}$.

Proposition 1.12. [2] *Let* $w = \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} \in \mathbb{C}^k$. *Then* $\Theta_a^*(w) = \sum_{i=1}^k c_i a_i$
.

Proof. Let $\{e_i\}_{i=1}^k$ be the standard orthonormal basis for $\mathbb{C}^k$ and $v \in V$.

Observe that $\langle \Theta_a^* e_i | v \rangle = \langle e_i | \Theta_a v \rangle = \left\langle \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \middle| \begin{pmatrix} \langle v|a_1\rangle \\ \vdots \\ \langle v|a_i\rangle \\ \vdots \\ \langle v|a_k\rangle \end{pmatrix} \right\rangle =$ $\overline{\langle v|a_i\rangle} = \langle a_i|v\rangle$. Since this is true for any $v \in V$, then $\Theta_a^*(e_i) = a_i$.

Now note that

$$\Theta_a^*(w) = \Theta_a^*(\sum_{i=1}^k c_i e_i) = \sum_{i=1}^k c_i \Theta_a^*(e_i) = \sum_{i=1}^k c_i a_i \ . \quad \square$$

Definition 1.13. Let $S = \Theta^*\Theta$. We call S the **frame operator**.

2. Orthonormal Vectors to Parseval Frames

Theorem 2.1. *Let $k \geq n$ and let A be an $n \times k$ matrix in which the rows form an orthonormal set of vectors in $\mathbb{R}^n$. Now let $F = \{v_1, \dots, v_k\}$ be the columns of A. Then F is a Parseval frame for $\mathbb{R}^n$.*

Proof. Let $A = [a_{ij}]$. We know

$$\langle x|v_j\rangle = \sum_{i=1}^n x_i a_{ij}.$$

Therefore we have

$$\begin{aligned}\sum_{j=1}^k |\langle x|v_j\rangle|^2 &= \sum_{j=1}^k \left(\sum_{i=1}^n x_i a_{ij}\right)^2 \\ &= \sum_{j=1}^k \sum_{i=1}^n \sum_{l=1}^n x_i a_{ij} x_l a_{lj} \\ &= \sum_{i=1}^n \sum_{l=1}^n x_i x_l \left(\sum_{j=1}^k a_{ij} a_{lj}\right)\end{aligned}$$

Now since the rows of A are orthonormal vectors, then

$$\sum_{j=1}^{k} a_{ij}a_{lj} = \begin{cases} 1 & \text{if } i = l \\ 0 & \text{if } i \neq l \end{cases}$$

So,

$$\begin{aligned} \sum_{j=1}^{k} |\langle x|v_j\rangle|^2 &= \sum_{i=1}^{n}\sum_{l=1}^{n} x_i x_l \left(\sum_{j=1}^{k} a_{ij}a_{lj}\right) \\ &= \sum_{i=1}^{n} x_i x_i \\ &= ||x||^2 \end{aligned}$$

□

Theorem 2.2. *Let $k \geq n$ and let A be an $n \times k$ matrix in which the rows form an orthonormal set of vectors in $\mathbb{C}^k$. Now let $F = \{v_1, \dots, v_k\}$ be the columns of A. Then F is a Parseval frame for $\mathbb{C}^n$.*

Proof. Let $A = [a_{ij}]$. We know

$$\langle x|v_j\rangle = \sum_{i=1}^{n} x_i \overline{a_{ij}} \quad \text{and also} \quad |z|^2 = z\overline{z}.$$

Therefore we have

$$\begin{aligned} \sum_{j=1}^{k} |\langle x|v_j\rangle|^2 &= \sum_{j=1}^{k} \left| \left(\sum_{i=1}^{n} x_i \overline{a_{ij}}\right) \right|^2 \\ &= \sum_{j=1}^{k} \left(\sum_{i=1}^{n} x_i \overline{a_{ij}}\right) \overline{\left(\sum_{l=1}^{n} x_l \overline{a_{lj}}\right)} \\ &= \sum_{j=1}^{k}\sum_{i=1}^{n} \left(x_i \overline{a_{ij}} \sum_{l=1}^{n} \overline{x_l} a_{lj}\right) = \sum_{j=1}^{k}\sum_{i=1}^{n}\sum_{l=1}^{n} x_i \overline{a_{ij}}\, \overline{x_l} a_{lj} \\ &= \sum_{i=1}^{n}\sum_{l=1}^{n} x_i \overline{x_l} \sum_{j=1}^{k} a_{lj}\overline{a_{ij}} = \sum_{i=1}^{n} x_i \overline{x_i} = ||x||^2. \end{aligned}$$

□

3. Harmonic Frames

Definition 3.1. Let the vector space be $\mathbb{C}^n$. A **harmonic frame** is a collection of vectors $\eta_0, \dots, \eta_{m-1}$ (where $m > n$) such that for

$0 \leq k \leq m-1$

$$\eta_k = \frac{1}{\sqrt{m}} \begin{bmatrix} w_1^k \\ \vdots \\ w_n^k \end{bmatrix} \text{ where } w_h = e^{i(\frac{h \times 2\pi}{m})}$$

Noting that w_h is an m^{th} root of unity.

Note that this is a frame for $\mathbb{C}^n$ since it is a spanning set in a finite dimensional vector space. In fact it is a Parseval frame.

Lemma 3.2. *For any geometric sequence,* $\{ra^i\}_{i=0}^{k-1}$ *for* $a, r \in \mathbb{C}, a \neq 0, 1, \sum_{i=0}^{k-1} ra^i = \frac{r(1-a^k)}{1-a}$.

Proof. We first observe that

$$\begin{aligned} \sum_{i=0}^{k-1} ra^i - \sum_{i=0}^{k} ra^i &= r + ra + \cdots + ra^{k-1} - (ra + ra^2 + \cdots + ra^k) \\ &= r + 0 + \cdots + 0 - ra^k \\ &= r - ra^k \, . \end{aligned}$$

But since

$$\sum_{i=0}^{k-1} ra^i - \sum_{i=0}^{k} ra^i = \sum_{i=0}^{k-1} ra^i - a\sum_{i=0}^{k-1} ra^i = (1-a)\sum_{i=0}^{k-1} ra^i = r - ra^k,$$

we see

$$\sum_{i=0}^{k-1} ra^i = \frac{r(1-a^k)}{1-a},$$

which concludes our proof. $\square$

Definition 3.3. [4] The discrete Fourier transform of a function $f = \begin{bmatrix} a_0 & \cdots & a_n \end{bmatrix}$, which we denote $\hat{f}$, is defined $\hat{f} = \frac{1}{\sqrt{N}} f \sum_{i=0}^{N-1} \omega_i$, where $\omega_0, ..., \omega_{N-1}$ denote the column vectors of the $N \times N$ Fourier matrix ${}_F^N\Omega$. For the entry in the kth row and lth column of this matrix, we say ${}_F^N\Omega_{k,l} := (\omega_k)_l = e^{\frac{ikl2\pi}{N}} = \omega_N^{kl}$.

Corollary 3.4. *In the case where* $m = n$, *the collection of vectors* $\eta_0, \ldots, \eta_{m-1}$ *is a Parseval frame.*

Proof. We first expand the matrix $\begin{bmatrix} \eta_0 & \eta_2 & \ldots & \eta_{m-1} \end{bmatrix}$ into

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{i\frac{2\pi}{m}1} & e^{i\frac{2\pi}{m}2} & \cdots & e^{i\frac{2\pi}{m}(m-1)} \\ 1 & e^{i\frac{2\pi}{m}2} & e^{i\frac{2\pi}{m}4} & \cdots & e^{i\frac{2\pi}{m}2(m-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{i\frac{2\pi}{m}(m-1)} & e^{i\frac{2\pi}{m}2(m-1)} & \cdots & e^{i\frac{2\pi}{m}(m-1)(m-1)} \end{bmatrix}$$

which is the matrix representation of the inverse discrete Fourier transform on $\mathbb{C}^n$. We now observe that $e^{\frac{-2\pi}{m}(k-1)(r-1)}$ represents the entry of this matrix in the kth row and rth column. Since

$$e^{\frac{i2\pi}{m}(k-1)(r-1)} = e^{\frac{i2\pi}{m}(r-1)(k-1)},$$

we see this matrix is symmetric by the definition of a symmetric matrix. To prove orthogonality of the columns of this matrix we see that for $p \neq v$, by Lemma 3.1,

$$\begin{aligned} \langle \eta_p | \eta_v \rangle &= \sum_{j=0}^{m-1} e^{\frac{i2\pi}{m}j(p+v)} \\ &= \frac{1 - e^{\frac{i2\pi}{m}m(p+v)}}{1 - e^{\frac{i2\pi}{m}(p+v)}} \\ &= \frac{1 - e^{i2\pi(p+v)}}{1 - e^{\frac{i2\pi}{m}(p+v)}} \\ &= \frac{1 - (\cos(2\pi(p+v)) + i\sin(2\pi(p+v)))}{1 - e^{\frac{i2\pi}{m}(p+v)}} \\ &= \frac{1-(1+0)}{1 - e^{\frac{i2\pi}{m}(p+v)}} \\ &= 0. \end{aligned}$$

Since our matrix is symmetric and we have shown the columns are orthogonal, we know the rows are orthogonal as well. By Theorems 2.1 and 2.2, $\eta_0, ..., \eta_{m-1}$ is a Parseval frame, which concludes our proof. $\square$

Theorem 3.5. *If* $\{\eta_k\}_{k=0}^{m-1}$ *is a harmonic frame then it is also a Parseval frame.*

Proof. Let $\{e_i\}_{i=1}^n$ be the standard basis in $\mathbb{C}^n$.

$$\begin{aligned}
\sum_{j=0}^{m-1} |\langle x|\eta_j\rangle|^2 &= \sum_{j=0}^{m-1} \langle x|\eta_j\rangle \overline{\langle x|\eta_j\rangle} = \sum_{j=0}^{m-1} \langle x|\eta_j\rangle\langle \eta_j|x\rangle \\
&= \sum_{j=0}^{m-1} \langle\langle \eta_j|x\rangle x|\eta_j\rangle = \sum_{j-0}^{m-1} \langle\langle \eta_j|x\rangle \left(\sum_{l=1}^{n} \langle x|e_l\rangle e_l\right) |\eta_j\rangle \\
&= \sum_{j=0}^{m-1}\sum_{l=0}^{n} \langle\langle \eta_j|x\rangle\langle x|e_l\rangle e_l|\eta_j\rangle \\
&= \sum_{j=0}^{m-1}\sum_{l=1}^{n} \langle\langle \left(\sum_{q=1}^{n} \langle \eta_j|e_q\rangle e_q\right) |x\rangle\langle x|e_l\rangle e_l|\eta_j\rangle \\
&= \sum_{j=0}^{m-1}\sum_{l=1}^{n}\sum_{q=1}^{n} \langle \eta_j|e_q\rangle\langle e_q|x\rangle\langle x|e_l\rangle\langle e_l|\eta_j\rangle
\end{aligned}$$

Now note that $\langle v|e_i\rangle$ is the i^{th} entry of v and $\langle e_i|v\rangle$ is the conjugate of the i^{th} entry.

Letting x_i be the i^{th} entry of x, then

$$\begin{aligned}
\sum_{j=0}^{m-1} |\langle x|\eta_j\rangle|^2 &= \sum_{j=0}^{m-1}\sum_{l=1}^{n}\sum_{q=1}^{n} \langle\langle x|e_l\rangle e_q|x\rangle\langle\langle \eta_j|e_q\rangle e_l|\eta_j\rangle \\
&= \sum_{j=0}^{m-1}\sum_{l=1}^{n}\sum_{q=1}^{n} \langle x_l e_q|x\rangle\langle \frac{1}{\sqrt{m}} w_q^j e_l|\eta_j\rangle \\
&= \sum_{j=0}^{m-1}\sum_{l=1}^{n}\sum_{q=1}^{n} x_l\langle e_q|x\rangle \frac{1}{\sqrt{m}} w_q^j \langle e_l|\eta_j\rangle
\end{aligned}$$

Now since $w_m^j = e^{i(j\frac{h\times 2\pi}{m})}$, then

$$\begin{aligned}\sum_{j=0}^{m-1} |\langle x|\eta_j\rangle|^2 &= \frac{1}{m}\sum_{j=0}^{m-1}\sum_{l=1}^{n}\sum_{q=1}^{n} x_l\overline{x_q}w_q^j\overline{w}_l^j \\ &= \frac{1}{m}\sum_{j=0}^{m-1}\sum_{l=1}^{n}\sum_{q=1}^{n} x_l\overline{x_q}e^{i(\frac{(jq2\pi)}{m})}e^{-i(\frac{(jl2\pi)}{m})} \\ &= \frac{1}{m}\sum_{j=0}^{m-1}\sum_{l=1}^{n}\sum_{q=1}^{n} x_l\overline{x_q}e^{i(\frac{(q-l)j2\pi)}{m})} \\ &= \frac{1}{m}\sum_{l=1}^{n}\sum_{q=1}^{n}\left(\sum_{j=0}^{m-1} x_l\overline{x_q}e^{i(\frac{(q-l)j2\pi}{m})}\right).\end{aligned}$$

If $q \neq l$, as in Corollary 3.4, we show that the geometric series

$$\sum_{j=0}^{m-1} e^{i(\frac{(q-l)j2\pi}{m})} \text{ sums to } 0.$$

Thus,

$$\begin{aligned}\frac{1}{m}\sum_{l=0}^{n}\sum_{q=1}^{n}(\sum_{j=1}^{m-1} x_l\overline{x_q}e^{i(\frac{(q-l)j2\pi}{m})}) &= \frac{1}{m}\sum_{l=0}^{n}\sum_{j=0}^{m-1} x_l\overline{x_l}e^{i(\frac{(0)2\pi}{m}j)} \\ &= \frac{1}{m}\sum_{l=1}^{n}(m)x_l\overline{x_l}(1) \\ &= \sum_{l=1}^{n} x_l\overline{x_l} \\ &= ||x||^2.\end{aligned}$$

□

4. Quasi-Gram-Schmidt Technique

The Gram-Schmidt Algorithm takes any basis and produces an orthonormal basis. The goal of the Quasi-Gram-Schmidt technique is to take any frame and produce a Parseval frame.

Definition 4.1. For T and R operators on a Hilbert space, if $T = R^2$, we say R is the square root of T, denoted by $T^{1/2}$.

Lemma 4.2. *If T is a positive operator on a Hilbert space, then T has a positive square root.*

Proof. By the definition of a positive operator and the Spectral Theorem, T has an orthonormal basis consisting of eigenvectors. If v_i is one of these eigenvectors of T corresponding to the eigenvalue λ_i then

$$0 \leq \langle Tv_i | v_i \rangle = \langle \lambda_i v_i | v_i \rangle = \lambda_i \langle v_i | v_i \rangle,$$

implying $\lambda_i \geq 0$. We can now define R such that $Rv_i = (\lambda_i)^{1/2} v_i$ since $\lambda_i \geq 0$. This implies that

$$\langle Rv_i | v_i \rangle = \langle (\lambda_i)^{1/2} v_i | v_i \rangle = (\lambda_i)^{1/2} \langle v_i | v_i \rangle \geq 0.$$

Additionally, since $((\lambda_i)^{1/2})^* = (\lambda_i)^{1/2}$,

$$\langle v_i | R^* v_i \rangle = \langle Rv_i | v_i \rangle = \langle (\lambda_i)^{1/2} v_i | v_i \rangle = \langle v_i | (\lambda_i)^{1/2} v_i \rangle,$$

implying $Rv_i = (\lambda_i)^{1/2} v_i = R^* v_i$. Since $\{v_i\}$ span the Hilbert space and $Rv_i = R^* v_i$ for all v_i, then $Rv = R^* v$ for any v in the Hlbert Space, and so $R = R^*$. Because R is self-adjoint and $\langle Rv_i | v_i \rangle \geq 0$, R is positive. Since

$$\begin{aligned} \langle v_i | R^2 v_i \rangle &= \langle v_i | R^* R v_i \rangle \\ &= \langle Rv_i | Rv_i \rangle \\ &= ||Rv_i||^2 \\ &= ||(\lambda_i)^{1/2} v_i||^2 \\ &= \langle (\lambda_i)^{1/2} v_i | (\lambda_i)^{1/2} v_i \rangle \\ &= \langle v_i | ((\lambda_i)^{1/2})^* (\lambda_i)^{1/2} v_i \rangle \\ &= \langle v_i | \lambda_i v_i \rangle, \end{aligned}$$

we see $R^2 e_i = \lambda_i e_i$. By substitution, $Tv_i = \lambda_i v_i = R^2 v_i$, implying $T = R^2$, which concludes our proof. □

Lemma 4.3. *If T is a positive operator on a Hilbert space V, assuming T is invertible, T^{-1} is also positive on V.*

Proof. By the properties of an adjoint, we know that since $T = T^*, T^{-1} = (T^*)^{-1} = (T^{-1})^*$, meaning T^{-1} is also self-adjoint. By Lemma 4.2, there exists a positive R on V such that $T = R^2$. Since T is invertible, if $v \in V$, we know $range(T) = V$. Therefore, $R(Rv) = Ru$ for some $u \in V$, and since we know $Tv = Ru$,

$$range(R) = range(T) = V,$$

meaning R is surjective. We now begin a proof by contradiction by assuming $null(R) \neq 0$. Therefore, $\exists g \in V$ such that $g \neq 0$ and $Rg = 0$. This means

$$Tg = RRg = R(0) = 0,$$

which contradicts the fact that $null(T) = \{0\}$. By contradiciton, $null(R) = \{0\}$, meaning R is injective and therefore bijective. This means R is invertible, and we denote its inverse by R^{-1}. By the definition of V if $v \in V$,

$$0 \leq \langle R^{-1}v | R^{-1}v \rangle = \langle (R^{-1})^2 v | v \rangle = \langle T^{-1}v | v \rangle,$$

which concludes our proof. □

Lemma 4.4. *Let A, D be complex square matrices, and let D be diagonal. If $A = PDP^{-1}$, where P is invertible, then $A^n = PD^nP^{-1}$ for all $n \in \mathbb{N}$.*

Proof. Since $A = PDP^{-1}$, by substitution,

$$\begin{aligned} A^n &= (PDP^{-1})^n \\ &= PDP^{-1}PDP^{-1} \cdots PDP^{-1}PDP^{-1} = PDIDI \cdots IDIDP^{-1} \\ &= PD^nP^{-1} \end{aligned}$$

which concludes our proof. □

This leads us to define A^α for non-integer α similarliy:

Definition 4.5. If $A = PDP^{-1}$, $A^\alpha = PD^\alpha P^{-1}$ for any $0 < \alpha \in \mathbb{R}$. For $A^{-\alpha}$, let $A^{-\alpha} = (A^{-1})^\alpha$. Finally, set $A^0 = I$.

Lemma 4.6. *Suppose A is diagonalizable. Then $(A^{-1})^\alpha = (A^\alpha)^{-1}$.*

Proof. Let $A = PDP^{-1}$, where D is a diagonal matrix.

$$\begin{aligned} (A^{-1})^\alpha &= ((PDP^{-1})^{-1})^\alpha \\ &= (P(PD)^{-1})^\alpha \\ &= (PD^{-1}P^{-1})^\alpha \\ &= PD^{-\alpha}P^{-1} \end{aligned}$$

Now,

$$\begin{aligned} (PD^\alpha P^{-1})(PD^{-\alpha}P^{-1}) &= PD^\alpha D^{-\alpha}P^{-1} \\ &= PP^{-1} \\ &= I. \end{aligned}$$

So,

$$(A^\alpha)^{-1} = (PD^\alpha P^{-1})^{-1}$$

which was just shown to be $PD^{-\alpha}P^{-1} = (A^{-1})^\alpha$. □

Lemma 4.7. *Let $S = \Theta^*\Theta$ be the frame operator.*

$$S^{-\frac{1}{2}^*} = S^{-\frac{1}{2}}$$

Proof. Since S is positive, $S^{\frac{1}{2}}$ is positive. So $S^{\frac{1}{2}}$ is self-adjoint. Thus,

$$S^{\frac{1}{2}^*} = S^{\frac{1}{2}}.$$

Now,

$$\begin{aligned} S^{-\frac{1}{2}^*} &= \left((S^{\frac{1}{2}})^{-1}\right)^* \\ &= ((S^{\frac{1}{2}})^*)^{-1} \\ &= (S^{\frac{1}{2}})^{-1} \\ &= S^{-\frac{1}{2}}. \end{aligned}$$

So $S^{-\frac{1}{2}}$ is self-adjoint. ☐

Theorem 4.8 (Quasi-Gram-Schmidt Theorem). *Let V be a finite k-dimensional inner product space. Let $\{v_i\}_{i=1}^n$ be a frame for V. Let $S = \Theta^*\Theta$. Then $\{S^{-\frac{1}{2}}v_i\}_{i=1}^n$ is a Parseval frame for V.*

Proof. Let $x \in v$, $S = \Theta^*\Theta$. Now S is self-adjoint, normal, and positive. Since S is normal, by the Spectral Theorem we can say V has an orthonormal basis of eigenvectors $\{e_1, \ldots, e_k\}$, such that $Se_i = \lambda_i e_i$ (where $\lambda_1, \ldots, \lambda_k$ are not necessarily distinct). Let M be a $k \times k$ matrix with columns e_i. So, $M = [e_1 \ldots e_k]$. (Note that $M^* = M^{-1}$)

Now,

$$\begin{aligned} SM &= [Se_1, \ldots, Se_k] \\ &= [\lambda_1 e_1, \ldots, \lambda_k e_k] \\ &= [e_1, \ldots, e_k] \begin{bmatrix} \lambda_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \lambda_k \end{bmatrix} \\ &= MD \end{aligned}$$

Where $D = \begin{bmatrix} \lambda_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \lambda_k \end{bmatrix}$
is a diagonal matrix. So $S = MDM^{-1}$, and S is diagonalizable.

Now,

$$\begin{aligned} \sum_{i=1}^n \left|\langle x|S^{-\frac{1}{2}}v_i\rangle\right|^2 &= \sum_{i=1}^n \langle x|S^{-\frac{1}{2}}v_i\rangle\langle S^{-\frac{1}{2}}v_i|x\rangle \\ &= \sum_{i=1}^n \langle x|S^{-\frac{1}{2}}v_i\rangle\langle S^{\frac{1}{2}}S^{-\frac{1}{2}}S^{-\frac{1}{2}}v_i|x\rangle \end{aligned}$$

By Lemma 4.7, ${S^{-\frac{1}{2}}}^* = S^{-\frac{1}{2}}$. Also note that $\{S^{-1}v_i\}$ is the canonical dual frame of $\{v_i\}$.

Thus we arrive at

$$\begin{aligned}\sum_{i=1}^{n}\langle S^{-\frac{1}{2}}x|v_i\rangle\langle S^{-1}v_i|S^{\frac{1}{2}}x\rangle &= \sum_{i=1}^{n}\langle\langle S^{\frac{1}{2}}x|v_i\rangle S^{-1}v_i|S^{\frac{1}{2}}x\rangle \\ &= \langle S^{-\frac{1}{2}}x|S^{\frac{1}{2}}x\rangle \\ &= \langle x|S^{-\frac{1}{2}}S^{\frac{1}{2}}x\rangle \\ &= \langle x|x\rangle \\ &= ||x||^2\end{aligned}$$

So, $\{S^{-\frac{1}{2}}v_i\}_{i=1}^n$ is a Parseval frame. □

Theorem 4.9. *Let $\{f_n\}_{n=1}^k$ be a frame for m-dimensional V. Let $S = \Theta^*\Theta$. Then*

$$S = [s_{ij}] = \left[\sum_{n=1}^{k} f_n[i]\overline{f_n[j]}\right]$$

where $f_n[k]$ is the k^{th} element of the vector f_n and s_{ij} is the element of S in the i^{th} row and j^{th} column.

Proof. Let $v \in V$. Then,

$$S(v) = \Theta^*\Theta(v) = \Theta^*\begin{pmatrix}\langle v|f_1\rangle \\ \vdots \\ \langle v|f_k\rangle\end{pmatrix} = \sum_{n=1}^{k}\langle v|f_n\rangle f_n\,.$$

Expanding the inner product we get,

$$S(v) = \sum_{n=1}^{k}\sum_{l=1}^{m}\left(v[l]\overline{f_n[l]}\right)f_n = \sum_{n=1}^{k}\sum_{l=1}^{m}\begin{pmatrix} v[l]\overline{f_n[l]}f_n[1] \\ \vdots \\ v[l]\overline{f_n[l]}f_n[m]\end{pmatrix}.$$

By vector addtion, when we sum over l we arrive at,

$$S(v) = \sum_{n=1}^{k} \begin{pmatrix} \sum_{l=1}^{m} v[l]\overline{f_n[l]}f_n[1] \\ \vdots \\ \sum_{l=1}^{m} v[l]\overline{f_n[l]}f_n[m] \end{pmatrix}$$
$$= \sum_{n=1}^{k} \begin{pmatrix} v[1]\overline{f_n[1]}f_n[1] & + \ldots + & v[m]\overline{f_n[m]}f_n[1] \\ & \vdots & \\ v[l]\overline{f_n[1]}f_1[m] & + \ldots + & v[m]\overline{f_n[m]}f_n[m] \end{pmatrix}.$$

By matrix multiplication we have,

$$S(v) = \sum_{n=1}^{k} \begin{bmatrix} \overline{f_n[1]}f_n[1] & \ldots & \overline{f_n[m]}f_n[1] \\ \vdots & \ddots & \vdots \\ \overline{f_n[1]}f_1[m] & \ldots & \overline{f_n[m]}f_n[m] \end{bmatrix} \begin{bmatrix} v[1] \\ \vdots \\ v[m] \end{bmatrix}$$
$$= \left(\sum_{n=1}^{k} \begin{bmatrix} \overline{f_n[1]}f_n[1] & \ldots & \overline{f_n[m]}f_n[1] \\ \vdots & \ddots & \vdots \\ \overline{f_n[1]}f_1[m] & \ldots & \overline{f_n[m]}f_n[m] \end{bmatrix} \right) \begin{bmatrix} v[1] \\ \vdots \\ v[m] \end{bmatrix}$$
$$= \begin{bmatrix} \sum_{n=1}^{k} \overline{f_n[1]}f_n[1] & \ldots & \sum_{n=1}^{k} \overline{f_n[m]}f_n[1] \\ \vdots & \ddots & \vdots \\ \sum_{n=1}^{k} \overline{f_n[1]}f_1[m] & \ldots & \sum_{n=1}^{k} \overline{f_n[m]}f_n[m] \end{bmatrix} \begin{bmatrix} v[1] \\ \vdots \\ v[m] \end{bmatrix}$$
$$= \left[\sum_{n=1}^{k} \overline{f_n[j]}f_n[i] \right] \begin{bmatrix} v[1] \\ \vdots \\ v[m] \end{bmatrix}.$$

Since this is true for all $v \in V$, then

$$S = [s_{ij}] = \left[\sum_{n=1}^{k} f_n[i]\overline{f_n[j]} \right]. \quad \square$$

Now by Theorem 4.9, we can write a formula for each entry of S. So we can write a program in Mathematica for any frame $\{v_i\}$ to create S by generating each entry of S. Then we can use Mathematica to make the set $\{S^{-\frac{1}{2}}v_i\}$ which is a proven Parseval frame. Combining this coding, we have a program to input a frame and recieve a Parseval frame:

```
frame = {}; (*Enter the frame set here.*)
a = ; (*Enter the number of vectors in the frame.*)
b = ; (*Enter the number of elements in a vector.*)

S = Table[Sum[frame[[n]][[i]] * Conjugate[frame[[n]][[j]]], {n, 1, a}], {i, b}, {j, b}]
Pframe = Table[MatrixPower[S, -0.5].frame[[i]], {i, a}]
```

To work the program, the frame set $\{v_i\}_{i=1}^k$ for $\mathbb{C}^n$ is inserted as "`frame =` $\{v_1, v_2, ..., v_k\}$". Then insert the number of vectors in the frame as "`a=` k". Then the number of elements in the individual vectors are inserted as "`b=` n". Then Mathematica generates the S frame operator in the fourth line. The fifth line generates the Parseval frame $\{S^{-\frac{1}{2}}v_i\}_{i=1}^k$ and labels the set as `Pframe`. To test that `Pframe` is truely a Parseval frame we can use the coding:

```
Expand[Sum[(Array[x, b].Conjugate[Pframe[[i]]]) * Pframe[[i]], {i, 1, a}]]
```

By Proposition 1.8, this checks that $\sum_{i=1}^k \langle x|v_i\rangle v_i = x$ for any

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{C}$$

proving that it is indeed a Parseval frame. Because Mathematica gives numerical approximations for the vectors in `Pframe`, there is some round-off error for the vectors which is usually detected by the test. For example, instead of the first entry of $\sum_{i=1}^k \langle x|v_i\rangle v_i = x$ equaling x_1 it would equal $1 \times x_1 + 2.345 \times 10^{-15} \times x_2 + ... + 4.7245 \times 10^{-16} \times x_n$. These values at the 10^{-15} level can be assumed to stand for zero.

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