Parseval Frame Construction

Nathan Bush, Meredith Caldwell, Trey Trampel

LSU, LSU, USA

July 6, 2012

1. Introduction

Introduction

A **vector space**, V, is a nonempty set with two operations: addition and multiplication by scalars such that the following conditions are satisfied for any $x, y, z \in V$ and any α, β in $\mathbb{R}$ and $\mathbb{C}$.[1]

$$x + y = y + x \qquad (1)$$

$$(x + y) + z = x + (y + z) \qquad (2)$$

$$x + z = y \text{ has a unique solution } z \text{ for each pair } (x, y) \qquad (3)$$

$$\alpha(\beta x) = (\alpha\beta)x \qquad (4)$$

$$(\alpha + \beta)x = \alpha x + \beta x \qquad (5)$$

$$\alpha(x + y) = \alpha x + \alpha y \qquad (6)$$

$$1x = x \qquad (7)$$

[1] Han, Kornelson, Larson, Weber

Introduction

The **dimension** of V is the number of elements contained in any basis.

Introduction

We say a complex (or real) vector space V is a **Hilbert space** if it is finite-dimensional and equipped with an inner product $\langle f|g\rangle$, that is, a map of $V \times V \to \mathbb{C}$ which satisfies

$$\langle f+g|h\rangle = \langle f|h\rangle + \langle g|h\rangle \tag{8}$$

$$\langle \alpha f|g\rangle = \alpha\langle f|g\rangle \tag{9}$$

$$\langle h|g\rangle = \overline{\langle g|h\rangle} \tag{10}$$

$$\langle f|f\rangle = 0 \text{ then } f = 0 \tag{11}$$

for all scalars α and f, g, h in V. For $x \in V$, we write $||x|| = \sqrt{\langle x|x\rangle}$, a nonnegative real number called the **norm** of x.

Introduction

We study **frames** in these Hilbert spaces, a generalization of the concept of a basis of a vector space.

Introduction

A **frame** for a Hilbert space V is a finite sequence of vectors $\{x_i\}_{i=1}^{k} \subset V$ for which there exist constants $0 < A \leq B < \infty$ such that, for every $x \in V$,

$$A||x||^2 \leq \sum_i |\langle x|x_i\rangle|^2 \leq B||x||^2.$$

A frame is a **Parseval frame** if $A = B = 1$. [2]

[2] Han, Kornelson, Larson, Weber

Introduction

Theorem

Suppose that V is a finite-dimensional Hilbert space and $\{x_i\}_{i=0}^{k}$ is a finite collection of vectors from V. Then the following statements are equivalent:[a]

- $\{x_i\}_{i=0}^{k}$ *is a frame for* V
- $span\,\{x_i\}_{i=0}^{k} = V$.

[a] Han, Kornelson, Larson, Weber

Introduction

Let $\{a_i\}_{i=1}^{k}$ be a finite sequence in V. Let $\Theta : V \to \mathbb{C}^k$ be defined as $\Theta(v) = \begin{pmatrix} \langle v|a_1\rangle \\ \vdots \\ \langle v|a_k\rangle \end{pmatrix}$ for $v \in V$. We call Θ the **analysis operator** for $\{a_i\}$.

By the Riesz Representation Theorem, every linear operator on V has an adjoint T^* such that $\langle Tx|y\rangle = \langle x|T^*y\rangle$. Let $\Theta^* : \mathbb{C}^k \to V$ be the adjoint of Θ. We call Θ^* the **reconstruction operator** for $\{a_i\}$.

Let $S = \Theta^*\Theta$. We call S the **frame operator**.

Introduction

Lemma

Reconstruction Formula Let $\{x_i\}_{i=1}^k$ *be a frame for a Hilbert space* V*. Then for every* $x \in V$*,*

$$x = \sum_{i=1}^{k} \langle x | S^{-1} x_i \rangle x_i = \sum_{i=1}^{k} \langle x | x_i \rangle S^{-1} x_i .$$

For this reason, $\{S^{-1}x_i\}$ is called the **canonical dual frame** of $\{x_i\}$.

Introduction

Lemma

Parseval Frame Reconstruction Formula A collection of vectors $\{x_i\}_{i=1}^{k}$ is a Parseval frame for a Hilbert space V if and only if the following formula holds for every x in V:

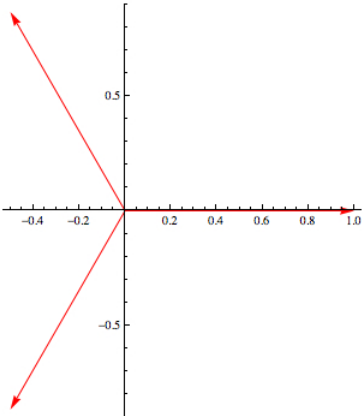
$$x = \sum_{i=1}^{k} \langle x | x_i \rangle x_i .$$

This equation is called the **reconstruction formula** *for a Parseval frame.*[a]

[a] Han, Kornelson, Larson, Weber

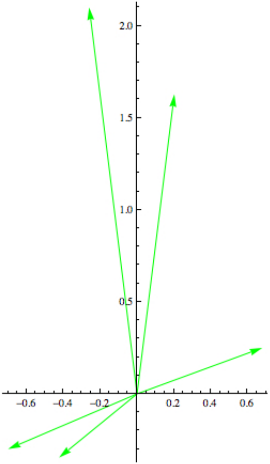
Examples of Frames

```
a = {1, 0}; b = {-1/2, Sqrt[3]/2}; c = {-1/2, -Sqrt[3]/2}
```



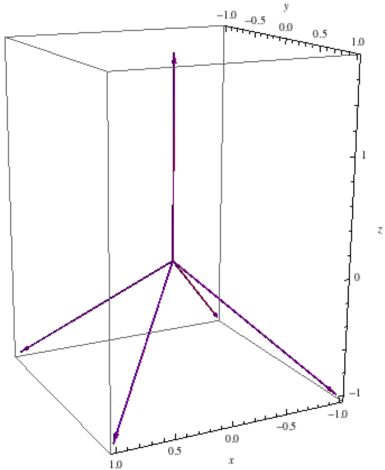
Examples of Frames

```
d = {.68, .25}; e = 88 / 100 {.23, 1.85}; f = {-.26, 2.1}; g = {-.7, -.3};
h = 77 / 100 {-.55, -.45}
```



Examples of Frames

```
i = {-1, -1, -1}; j = {1, -1, -1}; k = {-1, 1, -1}; l = {1, 1, -1};
m = {0, 0, Sqrt[3]}
```



Why are frames important?

Frames play an important role in signal processing, whether in transmitting images or sound waves. Parseval frames are especially important, as they allow for the reconstruction and interpretation of individual signals from a combination of many signals.

Orthonormal Vectors to Parseval Frames over $\mathbb{C}$ Theorem

Theorem

Let $k \geq n$ and let A be an $n \times k$ matrix in which the rows form an orthonormal set of vectors in $\mathbb{C}^k$. Now let $F = \{v_1, \ldots, v_k\}$ be the columns of A. Then F is a Parseval frame for $\mathbb{C}^n$.

Proof over $\mathbb{C}$

Let $A = [a_{ij}]$. We know

$$\langle x | v_j \rangle = \sum_{i=1}^{n} x_i \overline{a_{ij}} \quad \text{and} \quad |z|^2 = z\overline{z}$$

Therefore we have

$$\begin{aligned}
\sum_{j=1}^{k} |\langle x | v_j \rangle|^2 &= \sum_{j=1}^{k} \left| \left(\sum_{i=1}^{n} x_i \overline{a_{ij}} \right) \right|^2 \\
&= \sum_{j=1}^{k} \left(\sum_{i=1}^{n} x_i \overline{a_{ij}} \right) \overline{\left(\sum_{l=1}^{n} x_l \overline{a_{lj}} \right)} \\
&= \sum_{j=1}^{k} \sum_{i=1}^{n} \left(x_i \overline{a_{ij}} \sum_{l=1}^{n} \overline{x_l} a_{lj} \right)
\end{aligned}$$

Proof over $\mathbb{C}$

$$= \sum_{j=1}^{k} \sum_{i=1}^{n} \sum_{l=1}^{n} x_i \overline{a_{ij}}\, \overline{x_l} a_{lj} = \sum_{i=1}^{n} \sum_{l=1}^{n} x_i \overline{x_l} \sum_{j=1}^{k} a_{lj} \overline{a_{ij}}$$

Now since the rows of A are orthonormal vectors, then

$$\sum_{j=1}^{k} a_{ij} \overline{a_{lj}} = \begin{cases} 1 & \text{if } i = l \\ 0 & \text{if } i \neq l \end{cases}$$

So,

$$\sum_{j=1}^{k} |\langle x | v_j \rangle|^2 = \sum_{i=1}^{n} x_i \overline{x_i}$$

$$= ||x||^2. \quad \square$$

Harmonic Frames

Let the vector space be $\mathbb{C}^n$. A **harmonic frame** is a collection of vectors $\eta_0, \ldots, \eta_{m-1}$ (where $m \geq n$) such that for $0 \leq k \leq m-1$

$$\eta_k = \frac{1}{\sqrt{m}} \begin{bmatrix} w_1^k \\ \vdots \\ w_n^k \end{bmatrix}$$

where $w_h = e^{i\left(\frac{h \times 2\pi}{m}\right)}$ is an m^{th} root of unity. Note that this is a frame for $\mathbb{C}^n$ since it is a spanning set in a finite dimensional vector space.

Discrete Fourier Transform

When $m = n$, the matrix $\begin{bmatrix}\eta_0 & \dots & \eta_{m-1}\end{bmatrix}$ is the conjugate of the **Fourier matrix** of the discrete Fourier transform:

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{-i\frac{2\pi}{m}1} & e^{-i\frac{2\pi}{m}2} & \cdots & e^{-i\frac{2\pi}{m}(m-1)} \\ 1 & e^{-i\frac{2\pi}{m}2} & e^{-i\frac{2\pi}{m}4} & \cdots & e^{-i\frac{2\pi}{m}2(m-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-i\frac{2\pi}{m}(m-1)} & e^{-i\frac{2\pi}{m}2(m-1)} & \cdots & e^{-i\frac{2\pi}{m}(m-1)(m-1)} \end{bmatrix}$$

Geometric Series Lemma

Lemma

For any geometric sequence, $\{ra^i\}_{i=0}^{k-1}$ *where* $a, r \in \mathbb{C}, a \neq 0, 1,$

$$\sum_{i=0}^{k-1} ra^i = \frac{r(1-a^k)}{1-a}.$$

Geometric Series Proof

We first observe that

$$\begin{aligned}\sum_{i=0}^{k-1} ra^i - \sum_{i=0}^{k} ra^i &= r + ra + \cdots + ra^{k-1} - (ra + ra^2 + \cdots + ra^k) \\ &= r + 0 + \cdots + 0 - ra^k \\ &= r - ra^k\,.\end{aligned}$$

Geometric Series Proof

But since

$$\sum_{i=0}^{k-1} ra^i - \sum_{i=0}^{k} ra^i = \sum_{i=0}^{k-1} ra^i - a\sum_{i=0}^{k-1} ra^i = (1-a)\sum_{i=0}^{k-1} ra^i = r - ra^k,$$

we see

$$\sum_{i=0}^{k-1} ra^i = \frac{r(1-a^k)}{1-a},$$

which conclude our proof. $\square$

Harmonic Frames Theorem

Theorem

If $\{\eta_k\}_{k=0}^{m-1}$ is a harmonic frame then it is also a Parseval frame.

Harmonic Frames Proof

Let $\{e_i\}_{i=1}^{n}$ be the standard basis.

$$\sum_{j=0}^{m-1} |\langle x|\eta_j\rangle|^2 = \sum_{j=0}^{m-1} \langle x|\eta_j\rangle \overline{\langle x|\eta_j\rangle} = \sum_{j=0}^{m-1} \langle x|\eta_j\rangle\langle \eta_j|x\rangle$$

$$= \sum_{j-0}^{m-1} \langle \eta_j|x\rangle \langle \left(\sum_{l=1}^{n} \langle x|e_l\rangle e_l \right) |\eta_j\rangle$$

$$= \sum_{j=0}^{m-1} \sum_{l=1}^{n} \langle \left(\sum_{q=1}^{n} \langle \eta_j|e_q\rangle e_q \right) |x\rangle\langle x|e_l\rangle\langle e_l|\eta_j\rangle$$

$$= \sum_{j=0}^{m-1} \sum_{l=1}^{n} \sum_{q=1}^{n} \langle \eta_j|e_q\rangle\langle e_q|x\rangle\langle x|e_l\rangle\langle e_l|\eta_j\rangle$$

Harmonic Frames Proof

$$\sum_{j=0}^{m-1}\sum_{l=1}^{n}\sum_{q=1}^{n}\langle\eta_j|e_q\rangle\langle e_q|x\rangle\langle x|e_l\rangle\langle e_l|\eta_j\rangle = \sum_{j=0}^{m-1}\sum_{l=1}^{n}\sum_{q=1}^{n}\left(\frac{1}{\sqrt{m}}w_q^j\right)(\overline{x_q})(x_l)\left(\frac{1}{\sqrt{m}}\overline{w_l^j}\right)$$

$$= \frac{1}{m}\sum_{j=0}^{m-1}\sum_{l=1}^{n}\sum_{q=1}^{n}x_l\overline{x_q}w_q^j\overline{w_l^j}$$

Harmonic Frames Proof

$$= \frac{1}{m}\sum_{j=0}^{m-1}\sum_{l=1}^{n}\sum_{q=1}^{n} x_l\overline{x_q}e^{i(\frac{(jq2\pi)}{m})}e^{-i(\frac{(jl2\pi)}{m})}$$

$$= \frac{1}{m}\sum_{j=0}^{m-1}\sum_{l=1}^{n}\sum_{q=1}^{n} x_l\overline{x_q}e^{i(\frac{(q-l)j2\pi)}{m})}$$

$$= \frac{1}{m}\sum_{l=1}^{n}\sum_{q=1}^{n} x_l\overline{x_q}\left(\sum_{j=0}^{m-1} e^{i(\frac{(q-l)j2\pi}{m})}\right).$$

Harmonic Frames Proof

By our geometric series lemma, if $q \neq l$,

$$\begin{aligned}
\sum_{j=0}^{m-1} e^{i\left(\frac{(q-l)j2\pi}{m}\right)} &= \frac{1 - e^{i\left(\frac{(q-l)m2\pi}{m}\right)}}{1 - e^{i\left(\frac{(q-l)2\pi}{m}\right)}} \\
&= \frac{1 - e^{i((q-l)2\pi)}}{1 - e^{i\left(\frac{(q-l)2\pi}{m}\right)}} \\
&= \frac{1 - (\cos(2\pi(q-l)) + i\sin(2\pi(q-l)))}{1 - e^{i\left(\frac{(q-l)2\pi}{m}\right)}} \\
&= \frac{1 - (1+0)}{1 - e^{i\left(\frac{(q-l)2\pi}{m}\right)}} \\
&= 0.
\end{aligned}$$

Harmonic Frames Proof

Thus,

$$\begin{aligned}
\frac{1}{m}\sum_{l=1}^{n}\sum_{q=1}^{n} x_l\overline{x_q}\Big(\sum_{j=0}^{m-1} e^{i(\frac{(q-l)j2\pi}{m})}\Big) &= \frac{1}{m}\sum_{l=1}^{n} x_l\overline{x_l}\sum_{j=0}^{m-1} e^{i(\frac{(0)2\pi}{m}j)} \\
&= \frac{1}{m}\sum_{l=1}^{n} x_l\overline{x_l}(m) \\
&= \sum_{l=1}^{n} x_l\overline{x_l} \\
&= ||x||^2. \quad \square
\end{aligned}$$

Spectral Theorem

Theorem

Let T be a normal linear operator on a finite-dimensional Hilbert space V with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$, and let P_j be the orthogonal projection of V onto the eigenspace E_{λ_j} for $1 \leq j \leq k$. Then the following are true:[a]

$$\mathcal{V} = E_{\lambda 1} \oplus E_{\lambda 2} \oplus \ldots \oplus E_{\lambda k} \tag{12}$$

$$P_i P_j = 0 \text{ for } i \neq j \tag{13}$$

$$P_1 + P_2 + \ldots + P_k = I \tag{14}$$

$$T = \lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_k P_k \tag{15}$$

[a] Han, Kornelson, Larson, Weber

Spectral Theorem

Another statement of the Spectral Theorem is as follows:

Theorem

T is a normal linear operator on a finite dimensional Hilbert space if and only if there exists an orthonormal eigenbasis. [a]

[a] Axler

Lemma (9)

Lemma

Suppose S is diagonalizable. Then $(S^{-1})^{\frac{1}{2}} = (S^{\frac{1}{2}})^{-1}$.

Lemma (10)

Lemma

Let $S = \Theta^*\Theta$ be the frame operator. Then,

$$(S^{-\frac{1}{2}})^* = S^{-\frac{1}{2}}$$

Quasi Gram Schmidt Theorem

Theorem

Let V be a Hilbert space. Let $\{v_i\}_{i=1}^n$ be a frame for V. Let $S = \Theta^*\Theta$. Then $\{S^{-\frac{1}{2}}v_i\}_{i=1}^n$ is a Parseval frame for V.

Quasi Gram Schmidt Proof

Let $x \in V$, $S = \Theta^*\Theta$. Now S is self-adjoint, normal, and positive. Since S is normal, by the Spectral Theorem we can say V has an orthonormal basis of eigenvectors $\{e_1, \ldots, e_k\}$, such that $Se_i = \lambda_i e_i$ (where $\lambda_1, \ldots, \lambda_k$ are not necessarily distinct). Let M be a $k \times k$ matrix with columns e_i. So, $M = [e_1 \ldots e_k]$. (Note that $M^* = M^{-1}$)
Note that

$$\begin{aligned} SM &= [Se_1, \ldots, Se_k] \\ &= [\lambda_1 e_1, \ldots, \lambda_k e_k] \\ &= [e_1, \ldots, e_k] \begin{bmatrix} \lambda_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \lambda_k \end{bmatrix} \\ &= MD \end{aligned}$$

So, $S = MDM^{-1}$, and S is diagonalizable.

Quasi Gram Schmidt Proof

Now,

$$\sum_{i=1}^{n} \left|\langle x|S^{-\frac{1}{2}}v_i\rangle\right|^2 = \sum_{i=1}^{n}\langle x|S^{-\frac{1}{2}}v_i\rangle\langle S^{-\frac{1}{2}}v_i|x\rangle$$
$$= \sum_{i=1}^{n}\langle x|S^{-\frac{1}{2}}v_i\rangle\langle S^{\frac{1}{2}}S^{-\frac{1}{2}}S^{-\frac{1}{2}}v_i|x\rangle$$

By Lemma (10), $S^{-\frac{1}{2}^*} = S^{-\frac{1}{2}}$. So this is equal to

$$\sum_{i=1}^{n}\langle S^{-\frac{1}{2}}x|v_i\rangle\langle S^{-1}v_i|S^{\frac{1}{2}}x\rangle.$$

Quasi Gram Schmidt Proof

Now by Lemma (2),

$$\begin{aligned}\sum_{i=1}^{n}\langle S^{-\frac{1}{2}}x|v_i\rangle\langle S^{-1}v_i|S^{\frac{1}{2}}x\rangle &= \langle\sum_{i=1}^{n}\langle S^{-\frac{1}{2}}x|v_i\rangle S^{-1}v_i|S^{\frac{1}{2}}x\rangle \\ &= \langle S^{-\frac{1}{2}}x|S^{\frac{1}{2}}x\rangle \\ &= \langle x|S^{-\frac{1}{2}}S^{\frac{1}{2}}x\rangle \\ &= \langle x|x\rangle \\ &= ||x||^2 \quad \square\end{aligned}$$

So, $\{S^{-\frac{1}{2}}v_i\}_{i=1}^{n}$ is a Parseval frame.

Construction of Quasi Gram Schmidt Matrix Theorem

Theorem

Let $\{f_n\}_{n=1}^{k}$ be a frame for m-dimensional V. Let $S = \Theta^*\Theta$. Then

$$S = [s_{ij}] = \left[\sum_{n=1}^{k} f_n[i]\overline{f_n[j]}\right]$$

where $f_n[k]$ is the k^{th} element of the vector f_n and s_{ij} is the element of S in the i^{th} row and j^{th} column.

Program for forming a Parseval frame

Here is the Mathematica program for forming a Parseval frame for any frame in $\mathbb{C}^k$:

```
frame = {}; (*Enter the frame set here.*)
a = ; (*Enter the number of vectors in the frame.*)
b = ; (*Enter the number of elements in a vector.*)

S = Table[Sum[frame[[n]][[i]] * Conjugate[frame[[n]][[j]]], {n, 1, a}], {i, b}, {j, b}]
Pframe = Table[MatrixPower[S, -0.5].frame[[i]], {i, a}]
```

Program for forming a Parseval frame

To test that `Pframe` is truly a Parseval frame we can use the program:

```
Expand[Sum[(Array[x, b].Conjugate[Pframe[[i]]])*Pframe[[i]], {i, 1, a}]]
```

References

- Axler, Sheldon. *Linear Algebra Done Right*. Springer-New York, Inc. New York, NY, 1997.
- Davidson, Mark. *Frames*. SMILE@LSU. Louisiana State University, Baton Rouge. 4 - 27 June 2012. Lecture series.
- D. Han, K. Kornelson, D. Larson, E. Weber. *Frames for Undergraduates*. American Mathematical Society, Providence, RI, 2007.