1. Determine which sets of vectors form a basis for the given vector space:
   
   (a) $V = \mathbb{R}^3$, $B = \{(2, 1, -2), (-4, 2, 3), (1, 3, 5)\}$.
   
   (b) $V = \mathbb{R}^3$, $B = \{(2, 1, -2), (-4, 2, 3), (-8, 8, 5)\}$.
   
   (c) $V = \mathbb{R}^4$, $B = \{(0, 1, 1, 1), (0, 0, 1, 1), (1, 0, 0, 0)\}$.
   
   (d) $V = \mathcal{P}_3$, $B = \{1, 1 + x, 1 + x + x^2, 1 + x + x^2 + x^3\}$.

2. Find the coordinates of the vector $u$ with respect to the given basis:
   
   (a) $V = \mathbb{R}^3$, $u = (1, 0, 0)$, $B = \{(1, 1, 1), (-1, 1, 0), (1, 0, -1)\}$.
   
   (b) $V = \mathcal{P}_2$, $u = x^2 - 9$, $B = \{1, 1 + x, 1 + x + x^2\}$.

3. The vector space $\mathcal{P}_3$ has different bases such as:
   
   
   $B_1 = \{1, x, x^2, x^3\}$
   
   $B_2 = \left\{1, x - \frac{1}{2}, x^2 - x + \frac{1}{6}, x^3 - \frac{3}{2}x^2 + \frac{1}{2}x\right\}$

   The derivative $T = \frac{d}{dx}$ is a linear transformation from $\mathcal{P}_3$ into $\mathcal{P}_3$.

   (a) Find the matrix associated with $T$ relative to $B_1$ (i.e., find the matrix $A$ such that $\frac{d}{dx}p(x) = Ap(x)$ for $p \in \mathcal{P}_3$).
   
   (b) Find the matrix associated with $T$ relative to $B_2$.
   
   (c) Find $Ker(T)$ and $Im(T)$.

4. Determine which of the following sets form a vector space over $\mathbb{R}$.

   (a) The set of all the continuous functions $f : [-1, 1] \rightarrow \mathbb{R}$ that satisfy $f(-1) + f(1) = 0$.  

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(b) The set of all the continuous functions \( f : [-1, 1] \to \mathbb{R} \) that satisfy 
\[ f(-1) \cdot f(1) = 0. \]
(c) The set of all the continuous functions \( f : [-1, 1] \to \mathbb{R} \) that satisfy 
\[ \int_{-1}^{1} f(x) \, dx = 0. \]
(d) The set of all the continuous functions \( f : [-1, 1] \to \mathbb{R} \) that satisfy 
\[ \int_{-1}^{1} f^2(x) \, dx = 0. \]
(e) The set of all the polynomials \( p(x) \) with degree \( \deg(p) = 5 \).
(f) The set of all the polynomials \( p(x) \) with degree \( \deg(p) < 5 \).
(g) The set \( \{(x, y, z) \in \mathbb{R}^3 \mid x - 3z = 2\} \).

5. Consider \( S : \mathcal{P}_3 \to \mathbb{R}^3 \) given by
\[
S p = \left( \int_{-1}^{1} p(x) dx, p(0) + p'(0) + \frac{p''(0)}{2} + \frac{p'''(0)}{6}, p'(0) - 2p''(0) + p'''(0) \right)
\]
(a) Find the matrix of \( S \) relative to \( B = \{1, x, x^2, x^3\} \) and the canonical basis of \( \mathbb{R}^3 \).
(b) Solve the equation 
\[ Sp = (1, 1, 1). \]

6. Calculating a basis.
Let
\[
A = \begin{bmatrix} 2 & -2 \\ -1 & 1 \\ 1 & -1 \end{bmatrix}
\]
and \( T \in L(\mathbb{R}^2, \mathbb{R}^3) \) defined by
\[
T(x, y) = A \begin{bmatrix} x \\ y \end{bmatrix}
\]
Calculate a basis for \( Ker(T) \) and for \( Img(T) \).

7. Subspaces.
Let \( U = \{(x, y, z, w) \in \mathbb{R}^4 : w = 0, z = 5y - 2x\} \). Show that \( U \) is a subspace of \( \mathbb{R}^4 \), find a basis for \( U \) and show that it is in fact a basis.
8. Find the general solution of the following system of linear equations:

\[
\begin{align*}
    x + 2y + 4z &= 4 \\
    2x + 3y + 7z &= 3 \\
    4x + 5y + 13z &= 1
\end{align*}
\]

9. Solve (you might want to use Gauss-elimination method)

\[
Ax \equiv \begin{pmatrix}
    0 & 3 & 1 \\
    3 & 3 & 0 \\
    1 & 4 & 0
\end{pmatrix}
\begin{pmatrix}
    x_1 \\
    x_2 \\
    x_3
\end{pmatrix}
= \begin{pmatrix}
    5 \\
    9 \\
    6
\end{pmatrix}
\]

10. Find the inverse of

\[A = \begin{pmatrix}
    1 & 1 & 1 \\
    1 & 2 & 2 \\
    1 & 2 & 3
\end{pmatrix}\]

and then use it to solve

\[
A \begin{pmatrix}
    x \\
    y \\
    z
\end{pmatrix} = \begin{pmatrix}
    3 \\
    4 \\
    1
\end{pmatrix}
\]

To find the inverse of \(A\), write the augmented matrix

\[
\begin{pmatrix}
    1 & 1 & 1 & | & 1 & 0 & 0 \\
    1 & 2 & 2 & | & 0 & 1 & 0 \\
    1 & 2 & 3 & | & 0 & 0 & 1
\end{pmatrix}
\]

and then reduce it to the Guass-Jordan form.

11. Evaluate the determinant of the following matrices:

(a)

\[A = \begin{pmatrix}
    a & b \\
    c & d
\end{pmatrix}\]

(b)

\[B = \begin{pmatrix}
    1 & 2 & -1 \\
    0 & 2 & -1 \\
    0 & 2 & 7
\end{pmatrix}\]
12. Suppose we have four numbers \(a, b, c, d\) that were encoded to 1, 2, 3, and 4. The encoding algorithm is to multiply the vector \((a, b, c, d)^T\) by the matrix 
\[
G = \begin{pmatrix}
1 & -1 & 2 & 1 \\
0 & -1 & 5 & 3 \\
-1 & 2 & -3 & 0 \\
0 & 0 & 0 & 2
\end{pmatrix}
\]
to the left side. Try to decode it.

If we change the number field into \(\mathbb{F}_2\), and multiply \((a_1, a_2, \ldots, a_k)^T\) to the left side by a \(n \times k\) matrix, where \(k < n\), the encoding is linear coding.

13. Lagrange Polynomial Interpolation.
In engineering sometimes we want to find the exact form of a function. A polynomial is an excellent candidate. Given a set of points \((-2, 7), (-1, 4), (0, 1), (1, -2)\), find a polynomial of degree 4 that admits these points. In addition, try to give a general process of Lagrange Polynomial Interpolation with \(n\) points. Is the degree of the polynomial definitely \(n\)?

Find real numbers \(a, b, c\) so that the graph of the function \(y = ax^2 + bx + c\) contains the points \((-1, 4), (2, 3), (0, 1)\).

15. Least Square Fitting.
Given a set of points \((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\), find the best fitting straight line through a set of points. That means, find a straight line \(y = ax + b\), i.e., the value of \(a\) and \(b\), which minimizes \(V = \Sigma_{i=1}^{n} [y_i - (ax_i + b)]^2\). Try to use matrices to solve this problem.