Math Tune up Summer 2008

Exercises
Monday August 4

1. Determine which sets of vectors form a basis for the given vector space:

   (a) $V = \mathbb{R}^3$, $B = \{(2,1,-2), (-4,2,3), (1,3,5)\}$.

   (b) $V = \mathbb{R}^3$, $B = \{(2,1,-2), (-4,2,3), (-8,8,5)\}$.

   (c) $V = \mathbb{R}^4$, $B = \{(0,1,1,1), (0,0,1,1), (1,0,0,0)\}$.

   (d) $V = \mathcal{P}_3$, $B = \{1, 1+x, 1+x+x^2, 1+x+x^2+x^3\}$.

2. Find the coordinates of the vector u with respect to the given basis:

   (a) $V = \mathbb{R}^3$, $u = (1,0,0)$, $B = \{(1,1,1), (-1,1,0), (1,0,-1)\}$.

   (b) $V = \mathcal{P}_2$, $u = x^2 - 9$, $B = \{1, 1+x, 1+x+x^2\}$.

3. The vector space $\mathcal{P}_3$ has different bases such as:

$$\begin{aligned} B_1 &= \{1, x, x^2, x^3\} \\ B_2 &= \left\{1, x - \frac{1}{2}, x^2 - x + \frac{1}{6}, x^3 - \frac{3}{2}x^2 + \frac{1}{2}x\right\} \end{aligned}$$

   The derivative $T = \dfrac{d}{dx}$ is a linear transformation from $\mathcal{P}_3$ into $\mathcal{P}_3$.

   (a) Find the matrix associated with T relative to B_1 (i.e., find the matrix A such that $\frac{d}{dx}p(x) = Ap(x)$ for $p \in \mathcal{P}_3$).

   (b) Find the matrix associated with T relative to B_2.

   (c) Find $Ker(T)$ and $Im(T)$.

4. Determime which of the following sets form a vector space over $\mathbb{R}$.

   (a) The set of all the continuous functions $f : [-1,1] \to \mathbb{R}$ that satisfy $f(-1) + f(1) = 0$.

(b) The set of all the continuous functions $f : [-1, 1] \to \mathbb{R}$ that satisfy $f(-1) \cdot f(1) = 0$.

(c) The set of all the continuous functions $f : [-1, 1] \to \mathbb{R}$ that satisfy $\int_{-1}^{1} f(x)\, dx = 0$.

(d) The set of all the continuous functions $f : [-1, 1] \to \mathbb{R}$ that satisfy $\int_{-1}^{1} f^2(x)\, dx = 0$.

(e) The set of all the polynomials $p(x)$ with degree $deg(p) = 5$.

(f) The set of all the polynomials $p(x)$ with degree $deg(p) < 5$.

(g) The set $\{(x, y, z) \in \mathbb{R}^3 \mid x - 3z = 2\}$.

5. Consider $S : \mathcal{P}_3 \to \mathbb{R}^3$ given by

$$Sp = \left(\int_{-1}^{1} p(x)dx, p(0) + p'(0) + \frac{p''(0)}{2} + \frac{p'''(0)}{6}, p'(0) - 2p''(0) + p'''(0) \right)$$

(a) Find the matrix of S relative to $B = \{1, x, x^2, x^3\}$ and the canonical basis of $\mathbb{R}^3$.

(b) Solve the equation

$$Sp = (1, 1, 1) .$$

6. Calculating a basis.
Let

$$A = \begin{bmatrix} 2 & -2 \\ -1 & 1 \\ 1 & -1 \end{bmatrix}$$

and $T \in L(\mathbf{R}^2, \mathbf{R}^3)$ defined by

$$T(x, y) = A \begin{bmatrix} x \\ y \end{bmatrix}$$

Calculate a basis for $Ker(T)$ and for $Img(T)$.

7. Subspaces.
Let $U = \left\{(x, y, z, w) \in \mathbf{R}^4 : w = 0, z = 5y - 2x\right\}$. Show that U is a subspace of $\mathbf{R}^4$, find a basis for U and show that it is in fact a basis.

8. Find the general solution of the following system of linear equations:

$$\begin{array}{rcrcrcl} x & + & 2y & + & 4z & = & 4 \\ 2x & + & 3y & + & 7z & = & 3 \\ 4x & + & 5y & + & 13z & = & 1 \end{array}$$

9. Solve (you might want to use Gauss-elimination method)

$$Ax \equiv \begin{pmatrix} 0 & 3 & 1 \\ 3 & 3 & 0 \\ 1 & 4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 9 \\ 6 \end{pmatrix}$$

10. Find the inverse of

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

and then use it to solve

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$$

To find the inverse of A, write the augmented matrix

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right)$$

and then reduce it to the Guass-Jordan form.

11. Evaluate the determinant of the following matrices:

(a)

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(b)

$$B = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 2 & -1 \\ 0 & 2 & 7 \end{pmatrix}$$

(c)

$$C = \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & -1 & 5 & 3 \\ -1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

12. Suppose we have four numbers a, b, c, d that were encoded to $1, 2, 3$, and 4. The encoding algorithm is to multiply the vector $(a, b, c, d)^T$ by the matrix G=

$$\begin{pmatrix} 1 & -2 & -3 & -4 \\ 2 & 3 & -4 & -5 \\ -2 & -1 & 4 & 3 \\ 4 & 3 & -1 & -2 \end{pmatrix}$$

to the left side. Try to decode it.

If we change the number field into $\mathbb{F}_2$, and multiply $(a_1, a_2, \ldots, a_k)^T$ to the left side by a $n \times k$ matrix, where $k < n$, the encoding is linear coding.

13. Lagrange Polynomial Interpolation.
In engineering sometimes we want to find the exact form of a function. A polynomial is an excellent candidate. Given a set of points $(-2, 7), (-1, 4), (0, 1), (1, -2)$, find a polynomial of degree 4 that admits these points. In addition, try to give a general process of Lagrange Polynomial Interpolation with n points. Is the degree of the polynomial definitely n?

14. Curve Fitting.
Find real numbers a, b, and c so that the graph of the function $y = ax^2 + bx + c$ contains the points $(-1, 4)$,$(2, 3)$, and $(0, 1)$.

15. Least Square Fitting.
Given a set of points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, find the best fitting straight line through a set of points. That means, find a straight line $y = ax + b$, i.e., the value of a and b, which minimizes $V = \Sigma_{i=1}^{n}[y_i - (ax_i + b)]^2$. Try to use matrices to solve this problem.