FINDING PATTERNS IN THE POWERS OF A CHANGE OF BASIS MATRIX

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Abstract. Wilf's Conjecture states that there is no $n > 2$ for which the complementary Bell number of n, denoted $\tilde{B}(n)$, is zero. Here we study the complementary Bell numbers using a sequence of polynomials closely related to the complementary Bell numbers. Studying the complementary Bell numbers in this fashion naturally leads to the study of an infinite change of basis matrix, P. We introduce the concepts of Sirling numbers of the second kind, Bell and complementary Bell numbers, the matrix P, and we evaluate the structure of P^n for different values of n.

1. Introduction

The notation in this paper will follow that used in a course taught during the 2012 SMILE program at Louisiana State University by Dr. Valerio de Angelis from Xavier University of Louisiana. Lecture notes for this course make up the primary reference for the background information. Here we will be studying the sequence $\{\tilde{B}(n)\}_{n=1}^{\infty}$, the sequence of complementary Bell numbers, with the goal of further understanding Wilf's Conjecture. While studying the complementary Bell numbers, we find that

$$\tilde{B}(n+j) = \sum_{k=0}^{n} \lambda_j(k) S(n,k)$$

where $S(n,k)$ denotes the Stirling number of the second kind for n and k and λ_j is a polynomial of degree j satisfying

$$\lambda_{j+1}(x) = \lambda_j(x+1) - x\lambda_j(x).$$

We can then rewrite $\lambda_j(x)$ in terms of the basis $(x)_r$, the falling factorial, so that we have

$$\lambda_j(x) = \sum_{r=0}^{j} c_j(r)(x)_r.$$

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From here we find the infinite matrix P that is the focus of this paper. P is the matrix such that $c_{j+1} = Pc_j$, where c_j is the vector $(c_j(0), c_j(1), c_j(2), \dots)$. The matrix P has the following form:

$$\begin{aligned} &P(r,r) = r-1 \\ &P(r,r+1) = -(r+1) \\ &P(r+1,r) = 1 \\ &P(r,s) = 0 \quad \text{if } |r-s| > 1, \end{aligned}$$

where for convenience we take the convention that $r \geq 0$, meaning that the top-left entry of the matrix is denoted by $P(0,0)$.

This matrix is central to our approach to the study of the sequence $\tilde{B}(n)$. For instance, observe that

$$\tilde{B}(j) = \lambda_j(0) = c_j(0) = P^j(0,0).$$

Thus, by understanding powers of P, we can better understand $\tilde{B}(n)$. To begin this exploration, we introduce the concepts of the p-adic valuation, Stirling numbers of the second kind, and complementary Bell numbers.

2. Stirling numbers of the second kind and bell numbers

Stirling numbers of the second kind and Bell numbers are combinatorial objects that represent information regarding the number of partitions of a finite set. For our purposes, we will use $[n]$ to denote a finite set with n elements. A *partition* of a finite set $[n]$ is a collection of nonempty, disjoint sets whose union is $[n]$. Each of the sets making up a partition is called a *block*. Armed with this terminology, we are now ready to make the following definition.

Definition 2.1. We define the *Stirling numbers of the second kind*, denoted by $S(n,k)$, to be the number of ways to partition $[n]$ into k blocks, with $S(0,0)$ defined to be equal to 1.

For example, consider [3], a set with three elements, $\{a,b,c\}$. Then there are three ways to partition this set into two blocks. We could take $\{a\}$ and $\{b,c\}$, $\{a,b\}$ and $\{c\}$, or $\{a,c\}$ and $\{b\}$. Thus $S(3,2) = 3$.

Definition 2.2. We define the *Bell number* of n, denoted $B(n)$ to be the number of partitions of $[n]$, or

$$B(n) = \sum_{k=1}^{n} S(n,k).$$

In our example above, we found that $S(3,2) = 3$. There is only one way to partition [3] into 1 block, a, b, c, and also one way to partition [3] into 3 blocks, $\{a\}, \{b\}, \{c\}$. Thus $B(3) = S(3,1) + S(3,2) + S(3,3) = 1 + 3 + 1 = 5$.

Definition 2.3. The *complementary Bell number* of n, $\tilde{B}(n)$ is the number of partitions with an even number of blocks minus the number of partitions with an odd number of blocks, or

$$\tilde{B}(n) = \sum_{k=1}^{n} (-1)^k S(n,k).$$

Once again returning to our example, we found that $S(3,2)$ was 3, so there were three ways in total of partitioning [3] into an even number of blocks. There was one way to partition [3] into one block, and another with three blocks, so there were two ways to partition [3] into an odd number of blocks. Hence we see that

$$\tilde{B}(3) = S(3,2) - (S(3,1) + S(3,3)) = 3 - (1+1) = 1.$$

For $n = 2$, it is easy to verify that there is only one partition with an even number and one with an odd number, meaning that $\tilde{B}(2) = 0$. This leads us to a conjecture due to Herbert Wilf.

Conjecture 2.4 (Wilf's Conjecture)**.** $\tilde{B}(n) = 0$ *only for* $n = 2$.

This problem remains, as of the writing of this paper, unsolved. It has been shown that there is at most one other n for which $\tilde{B}(n) = 0$, but it remains to be seen whether or not this n exists.

It is easy to verify that for the Stirling numbers of the second kind we have the following recurrence:

$$S(n+1,k) = S(n,k-1) + kS(n,k).$$

Using this information, we can make the following proposition.

Proposition 2.5. $\tilde{B}(n+j) = \sum_{k=0}^{n+j}(-1)^k S(n+j,k) = \sum_{k=0}^{n}(-1)^k \lambda_j(k) S(n,k)$, *where* $\lambda_j(x)$ *are polynomials of degree* j *defined recursively by*

- $\lambda_0(k) = 1$
- $\lambda_{j+1} = x\lambda_j(x) - \lambda(x+1)$

Proof. We proceed by induction. First, assume $j = 1$. Then we have

$$\begin{aligned}
\tilde{B}(n) &= \sum_{k=0}^{n+1} (-1)^k S(n+, k) \\
&= \sum_{k=0}^{n+1} (-1)^k S(n, k-1) + \sum_{k=0}^{n+1} (-1)^k k S(n, k) \\
&= \sum_{k=1}^{n+2} (-1)^{k+1} S(n, k) + \sum_{k=0}^{n+1} (-1)^k k S(n, k) \\
&= \sum_{k=0}^{n} (-1)^{k+1} S(n, k) + \sum_{k=0}^{n} (-1)^k k S(n, k) \\
&= \sum_{k=0}^{n} (-1)^k (k-1) S(n, k) \\
&= \sum_{k=0}^{n} (-1)^k \lambda_1(k) S(n, k).
\end{aligned}$$

Now suppose that the claim holds for some $j \in \mathbb{N}$. Then, for $j+1$ we have

$$\begin{aligned}
\sum_{k=0}^{n+j+1} (-1)^k S(n+j+1, k) &= \sum_{k=0}^{(n+1)+j} (-1)^k S(n+j+1, k) \\
&= \sum_{k=0}^{n+1} (-1)^k \lambda_j(k) S(n+1, k) \\
&= \sum_{k=0}^{n+1} (-1)^k \lambda_j(k) (S(n, k-1) + k S(n, k)) \\
&= \sum_{k=0}^{n} (-1)^{k+1} \lambda_j(k+1) S(n, k) + \sum_{k=0}^{n} (-1)^k k \lambda_j(k) S(n, k) \\
&= \sum_{k=0}^{n} (-1)^k (k \lambda_j(k) - \lambda_j(k+1)) S(n, k) \\
&= \sum_{k=0}^{n} (-1)^k \lambda_{j+1}(k) S(n, k).
\end{aligned}$$

$\square$

Corollary 2.6. *For all $j \in \mathbb{N}$, we have that $\tilde{B}(j) = \lambda_j(0)$.*

We now have a way of expressing $\tilde{B}(n)$ in terms of these polynomials, λ_n. By rewriting these polynomials using the falling factorial as a basis, we can now prove the following theorem.

Theorem 2.7. *Let $P = P(r, s)$ where $r, s \geq 0$ be the infinite matrix defined by*

$$P(r,r) = r - 1 \qquad P(r, r+1) = -(r+1)$$
$$P(r+1, r) = 1 \qquad P(r,s) = 0 \text{ for } |r - s| > 1.$$

Then we have $\tilde{B}(n) = P^n(0, 0)$.

Proof. Rewriting $\lambda_n(k)$ in terms of the falling factorial, we get

$$\lambda_n(k) = \sum_{r=0}^{n} c_n(r)(k)_r.$$

From the previous proposition, we know that $c_0(0) = 1$, $c_0(r) = 0$ for all $r > 0$ and also we have that $c_n(r) = 0$ if $r > n$. Also, if we let $c_n = (c_n(0), c_n(1), c_n(3), \dots)$, we see that

$$c_{n+1} = Pc_n.$$

Since $\lambda_n(0) = c_n(0) = P^n(0, 0)$, the result follows. □

The previous theorem highlights the significance of this matrix P. We will now further explore the structure of P^n.

3. Powers of The Matrix P

The matrix P, as defined in Theorem 2.7 , has the following form:

$$\begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & -2 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & -3 & 0 & 0 & \dots \\ 0 & 0 & 1 & 2 & -4 & 0 & \dots \\ 0 & 0 & 0 & 1 & 3 & -5 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Using this information, as well as the definition in Theorem 2.7, we obtain the following result.

Theorem 3.1. *The matrix P^2 has the following form for all $r, s \geq 0$:*

- $P^2(r, r+2) = (r+1)(r+2)$
- $P^2(r, r+1) = -2r^2 - r + 1$
- $P^2(r, r) = r^2 - 4r$
- $P^2(r+1, r) = 2r - 1$
- $P^2(r+2, r) = 1$
- $P^2(r, s) = 0$ *if* $|r - s| > 2$

Proof. For $P^2(r,r)$, we have

$$\begin{aligned}
P^2(r,r) &= \sum_{k=0}^{\infty} P(r,k)P(k,r) \\
&= P(r,r-1)P(r-1,r) + P(r,r)^2 + P(r,r+1)P(r+1,r) \\
&= 1(-(r-1)-1) + (r-1)^2 + (-r-1)1 \\
&= -r + r^2 - 2r + 1 + r - 1 \\
&= r^2 - 4r.
\end{aligned}$$

In the case that we have $P^2(r,r+1)$, we see that

$$\begin{aligned}
P^2(r,r+1) &= \sum_{k=0}^{\infty} P(r,k)P(k,r+1) \\
&= P(r,r)P(r,r+1) + P(r,r+1)P(r+1,r+1) \\
&= (r-1)(-r-1) + (-r-1)(r) \\
&= -r^2 + 1 - r^2 - r \\
&= -2r^2 - r + 1.
\end{aligned}$$

When we consider $P^2(r,r+2)$, we see that

$$\begin{aligned}
P^2(r,r+2) &= \sum_{k=0}^{\infty} P(r,k)P(k,r+2) \\
&= P(r,r+1)P(r+1,r+2) \\
&= (-r-1)(-(r+1)-1) \\
&= (-r-1)(-r-2) \\
&= (r+1)(r+2).
\end{aligned}$$

Looking at $P^2(r+1,r)$, we get

$$\begin{aligned}
P^2(r+1,r) &= \sum_{k=0}^{\infty} P(r+1,k)P(k,r) \\
&= P(r+1,r)P(r,r) + P(r+1,r+1)P(r+1,r) \\
&= 1(r-1) + r(1) \\
&= r - 1 + r \\
&= 2r - 1.
\end{aligned}$$

Lastly, consider the case of $P^2(r+2,r)$:

$$\begin{aligned} P^2(r+2,r) &= \sum_{k=0}^{\infty} P(r+2,k)P(k,r) \\ &= P(r+2,r+1)P(r+1,r) \\ &= (1)(1) \\ &= 1 \end{aligned}$$

□

Using the same process as in Theorem 3.1, we are able to also obtain the next two theorems.

Theorem 3.2. *The matrix P^3 has the following form:*

- $P^3(r,r+3) = -(r+1)(r+2)(r+3)$
- $P^3(r,r+2) = 3(r)(r+1)(r+2)$
- $P^3(r,r+1) = -(r+1)(3r^2-6r-2)$
- $P^3(r,r) = r^3-9r^2+6r+1$
- $P^3(r+1,r) = 3r^2-6r-2$
- $P^3(r+2,r) = 3r$
- $P^3(r+3,r) = 1$
- $P^3(r,s) = 0$ *if* $|r-s| > 3$

Theorem 3.3. *The matrix P^4 has the following form:*

- $P^4(r,r+4) = (r+1)(r+2)(r+3)(r+4)$
- $P^4(r,r+3) = -2(r+1)(r+2)(r+3)(2r+1)$
- $P^4(r,r+2) = (r+1)(r+2)(6r^2-4r-5)$
- $P^4(r,r+1) = -(r+1)(4r^3-18r^2-2r+1)$
- $P^4(r,r) = r^4-16r^3+30r^2+ -4r+1$
- $P^4(r+1,r) = 4r^3-18r^2-2r+1$
- $P^4(r+2,r) = 6r^2-4r-5$
- $P^4(r+3,r) = 4r+2$
- $P^4(r+4,r) = 1$
- $P^4(r,s) = 0$ *if* $|r-s| > 4$

From what we have here, we are able to deduce some properties of P^n.

Theorem 3.4. $P^j(r,s) = 0$ *if* $|r-s| > j$ *for all* $j \in \mathbb{N}$

Proof. By the definition of P, $P^1(r,s) = 0$ if $|r-s| > 1$

Thus the base case, $j = 1$, holds. Now let us assume that this is true for some $j \in \mathbb{N}$ and assume that $|r-s| > j+1$. Then

$$\begin{aligned} P^{j+1}(r,s) &= \sum_{k=0}^{\infty} P^j(r,k)P(k,s) \\ &= P^j(r,s-1)P(s-1,s) + P^j(r,s)P(s,s) + P^j(r,s+1)P(s+1,s). \end{aligned}$$

Observe that $|r-s+1| = |r-s-(-1)| \geq |r-s|-|-1| > j+1-1 = j$, meaning that $P^j(r, s-1) = 0$. Also note that $|r-s| \geq j+1 > j$, so $P^j(r, s) = 0$. And lastly, $|r-s-1| \geq |r-s|-1 > j+1-1 = j$, so $P^j(r, s+1) = 0$. This yields

$$\begin{aligned} P^{j+1}(r, s) &= P^j(r, s-1)P(s-1, s) + P^j(r, s)P(s, s) + P^j(r, s+1)P(s+1, s) \\ &= 0+0+0 \\ &= 0. \end{aligned}$$

□

Theorem 3.5. *$P^j(r+j, r) = 1$ for all $j \in \mathbb{N}$.*

Proof. This is true for the base case, $j = 1$, from the definition of P. Now let us assume that this is true for $P^j(r+j, r)$. We know from the definition of matrix multiplication that

$$P^{j+1}(r+j+1, r) = \sum_{k=0}^{\infty} P^j(r+j+1, k)P^1(k, r),$$

and we know from the definition that $P^1_m(r, s) = 0$ if $|r-s| > 1$, meaning that the only three terms from our sum can be non-zero. Namely

$$\begin{aligned} P^{j+1}(r+j+1, r) &= \sum_{k=0}^{\infty} P^j(r+j+1, k)P^1(k, r) \\ &= P^j(r+j+1, r-1)P^1(r-1, r) + P^j(r+j+1, r)P^1(r, r) \\ &+ P^j(r+j+1, r+1)P^1(r+1, r) \end{aligned}$$

Note that

$$|(r+j+1-(r-1)| = |r+j+1-r+1| = |j+2| > j,$$

so by Theorem 3.4

$$P^j(r+j+1, r-1) = 0.$$

Similarly, we see that

$$|r+j+1-(r)| = |r+j+1-r| = |j+1| > j,$$

So,

$$P^j(r+j+1, r) = 0.$$

The last possible non-zero term is $P^j(r+j+1, r+1)P^1(r+1, r)$. Notice that $P^j(r+j+1, r+1) = 1$ by hypothesis, and $P(r+1, r) = 1$ by definition. Thus we have

$$P^{j+1}(r+j+1, r) = 1 * 1 = 1,$$

and the theorem is proved. □

Theorem 3.6. *$P^j(r, r+j) = (-1)^j(r+1)^{(j)}$ for all $j \in \mathbb{N}$, where $(x)^{(j)}$ denotes the rising factorial.*

Proof. For the case that $j = 1$, observe that $P(r, r+1) = -(r+1)$, and the claim holds. Suppose, then, that $P^j(r, r+j) = (-1)^j(r+1)^{(j)}$ for some $j \in \mathbb{N}$. Then we have

$$\begin{aligned}
P^{j+1}(r, r+j+1) &= \sum_{k=0}^{\infty} P^j(r, k)P(r+j, r+j+1) \\
&= P^j(r, r+j)P(r+j, r+j+1) \\
&(\text{ by Theorem 3.4 and definition of } P) \\
&= (-1)^j(r+1)^{(j)}(-r-j-1) \\
&= (-1)^{j+1}(r+1)^{(j)}(r+j+1) \\
&= (-1)^{j+1}(r+1)^{(j+1)}
\end{aligned}$$

□

There is also a connection between $P^j(r, 0)$ and $P^j(0, r)$, and in more general terms between $P^j(r, s)$ and $P^j(s, r)$.

Theorem 3.7. *$(-1)^{r-s}r!P^j(r, s) = s!P^j(s, r)$ for all $r, s \geq 0$.*

Proof. Let us first consider the base case, $j = 1$. There are four different scenarios. In the case where $|r - s| > 1$, we have $P(r, s) = P(s, r) = 0$, so this scenario is trivial. The case where we have $P(r, r)$ is also trivial. It is also possible that we have $P(r, r+1)$, which is equal to $-r - 1 = \frac{(-1)^{r-r+1}(r+1)!}{(r)!}P(r+1, r)$. Lastly, we have $P(r+1, r) = 1 = \frac{(-1)^{r-(r+1)}r!}{(r+1)!}P(r, r+1) = 1$. So the relation holds in the base case.

Now assume that $\frac{(-1)^{r-s}r!}{s!}P^j(r,s) = P^j(s,r)$ for some $j \in \mathbb{N}$. Then we have

$$\begin{aligned}
P^{j+1}(s,r) &= (P^jP)(s,r)\\
&= \sum_{k=0}^{\infty} P^j(s,k)P(k,r)\\
&= P^j(s,r-1)P(r-1,r) + P^j(s,r)P(r,r) + P^j(s,r+1)P(r+1,r)\\
&= -rP^j(s,r-1) + (r-1)P^j(s,r) + 1 \times P^j(s,r+1)\\
&= -r\frac{(-1)^{r-s-1}(r-1)!}{s!}P^j(r-1,s) + (r-1)\frac{(-1)^{r-s}r!}{s!}P^j(r,s)\\
&\quad + \frac{(-1)^{r-s+1}(r+1)!}{s!}P^j(r+1,s)\\
&= \frac{(-1)^{r-s}r!}{s!}P^j(r-1,s) + (r-1)\frac{(-1)^{r-s}r!}{s!}P^j(r,s)\\
&\quad + (-r-1)\frac{(-1)^{r-s}r!}{s!}P^j(r+1,s)\\
&= \frac{(-1)^{r-s}r!}{s!}[1 \times P^j(r-1,s) + (r-1)P^j(r,s) + (-r-1)P^j(r+1,s)]\\
&= \frac{(-1)^{r-s}r!}{s!}[P(r,r-1)P^j(r-1,s) + P(r,r)P^j(r,s)\\
&\quad + P(r,r+1)P^j(r+1,s)]\\
&= \frac{(-1)^{r-s}r!}{s!}(PP^j)(r,s)\\
&= \frac{(-1)^{r-s}r!}{s!}P^{j+1}(r,s).
\end{aligned}$$

Multiplying both sides by $s!$ yields the result. $\square$

4. Conclusions and Future Work

While the matrix P is infinite, which poses some problems, the structure is quite regular, with most entries being zero. We were able to find a few patterns in general and classify all of the structure for powers of P up to 4. Looking for more general patterns in the structure of powers of P could lead to more results and a further understanding of Wilf's Conjecture, such as finding an analog to Dr. de Angelis's Fundamental Structure Lemma for P modulo $3 * 2^n$ for different values of n.

Throughout the SMILE program, we used the 2-adic valuation to study Wilf's Conjecture. While time restraints kept us from exploring this option, looking at the 2-adic valuation of these matrices may also yield exciting results.

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