# The Complementary Bell Numbers

## Explored via a Matrix Constructed with Rising Factorials

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Note that both $(x)_r$ and $x^{(r)}$ are polynomials of degree $r$.

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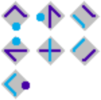

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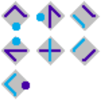

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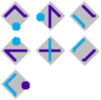

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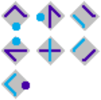


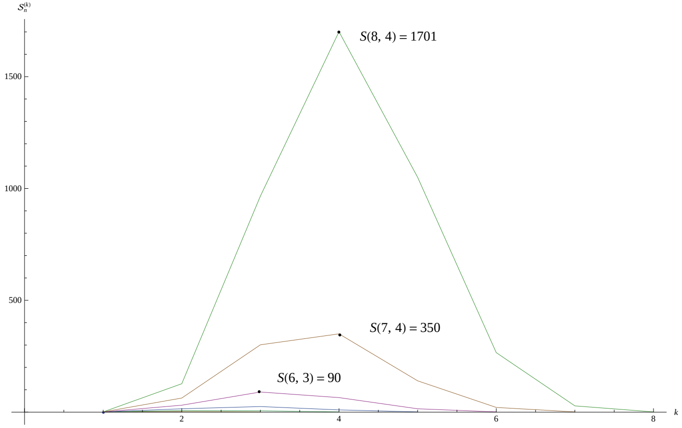
Figure: $S(4, 2) = 7$

Figure: $S(4, 3) = 6$

Figure: $S(4, 4) = 1$

Note: From the examples, it is clear that $S(n, 1) = S(n, n) = 1$.

# Growth




The points labeled are the $k$ values that yield the maximum $S(n, k)$ for a given $n$.

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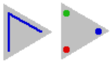
|  |  |

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$\widetilde{B}(4) = 1$

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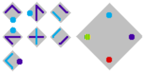
|  |  |

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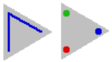
|  |  |

$\widetilde{B}(4) = 1$

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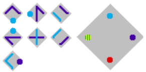
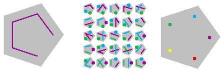
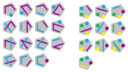
|  |  |

$\widetilde{B}(5) = -2$

| Odd | Even |
|---|---|
|  |  |

# Complementary Bell Numbers

| $n$ | $\widetilde{B}(n)$ |
|---|---|
| 0 | 1 |
| 1 | −1 |
| 2 | 0 |
| 3 | 1 |
| 4 | 1 |
| 5 | −2 |
| 6 | −9 |
| 7 | −9 |
| 8 | 50 |
| 9 | 267 |
| 10 | 413 |
| 11 | −2180 |
| 12 | −17731 |
| 13 | −50533 |
| 14 | 110176 |
| 15 | 1966797 |
| 16 | 9938669 |
| 17 | 8638718 |
| ⋮ | ⋮ |

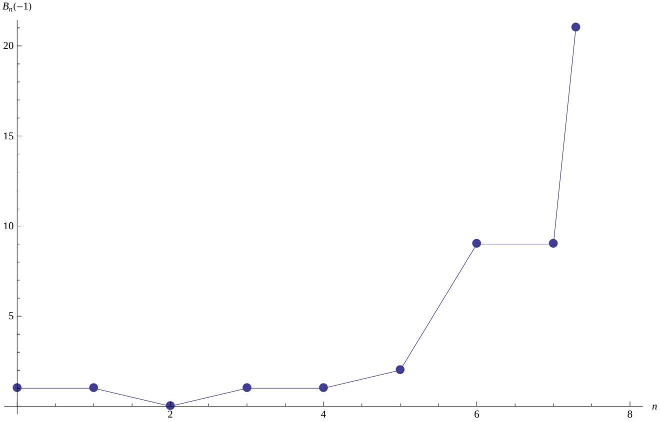



Figure: $|\widetilde{B}(n)|$ for $n \leq 8$

# Wilf's Conjecture

H.S. Wilf's Conjecture:

$$\widetilde{B}(n) \neq 0 \text{ for all } n > 2$$

# The $\lambda_j(x)$ Polynomials

There exist polynomials $\lambda_j$, for all $n, j \geq 0$, that satisfy

$$\widetilde{B}(n+j) = \sum_{k=0}^{n} (-1)^k \lambda_j(k) S(n,k)$$

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Note that $\lambda_n(x)$ is a monic polynomial of degree $n$.

# Rising Factorials as a Basis for $\mathcal{P}_n$

**Theorem**

*For each $n \geq 0$, the set of rising factorials $\left\{x^{(k)} : 0 \leq k \leq n\right\}$ is a basis for $\mathcal{P}_n$, the vector space of polynomials of degree less than or equal to $n$.*

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$$x^n = \sum_{k=0}^{n} (-1)^{n+k} S(n,k) x^{(k)} \text{ for all } n \geq 0$$

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By the previous theorem:

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Therefore:

$$\sum_{k=0}^{n+1} a_{n+1}(k)\,(x-1)^{(k)} = \sum_{k=0}^{n} a_n(k)\,(x-1)\,(x-1)^{(k)} - \sum_{k=0}^{n} a_n(k)x^{(k)}$$

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**Lemma**

*For all $n \geq 0$ and for all $0 \leq k \leq n+1$,*

$$a_{n+1}(k) - (k+1)\,a_{n+1}(k+1) = a_n(k-1) - 2\,(k+1)\,a_n(k) + (k+1)^2\,a_n(k+1)$$

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$$A^{-1} = \begin{pmatrix} 1 & 1 & 2 & 6 & 24 & \dots \\ 0 & 1 & 2 & 6 & 24 & \dots \\ 0 & 0 & 1 & 3 & 12 & \dots \\ 0 & 0 & 0 & 1 & 4 & \dots \\ 0 & 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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$$R(i,j) = \begin{cases} -\frac{j!}{i!} & \text{if } j > i \\ -(i+1) & \text{if } j = i \\ 1 & \text{if } j = i-1 \\ 0 & \text{if } j < i-1 \end{cases}$$

# The $R$-Matrix

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# The Lower Section

**Lemma**

*For each* $n \in \mathbb{N}$, *the nth power of* $R$ *is defined and* $R^n(i,j) = 0$ *if* $j < i - n$.

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$$R^2 = \begin{pmatrix} 0 & 1 & 4 & 18 & 96 & \cdots \\ -3 & 1 & 2 & 12 & 72 & \cdots \\ 1 & -5 & 4 & 3 & 24 & \cdots \\ 0 & 1 & -7 & 9 & 4 & \cdots \\ 0 & 0 & 1 & -9 & 16 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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$$R^3 = \begin{pmatrix} 1 & 2 & 4 & 6 & -24 & \cdots \\ 4 & 3 & 10 & 30 & 96 & \cdots \\ -6 & 13 & -1 & 24 & 96 & \cdots \\ 1 & -9 & 28 & -17 & 44 & \cdots \\ 0 & 1 & -12 & 49 & -51 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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# The Top Row

**Lemma**

*For all $n \geq 1$ and $j \geq 0$, the $(0, j)$th entry of $R^n$ is divisible by $j!$.*

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| $j$ | $j!$ |
|---|---|
| 0 | 1 |
| 1 | 1 |
| 2 | 2 |
| 3 | 6 |
| 4 | 24 |
| 5 | 120 |
| 6 | 720 |
| $\vdots$ | $\vdots$ |

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$$R = \begin{pmatrix} -1 & -1 & -2 & -6 & -24 & -120 & -720 & \dots \\ 1 & -2 & -2 & -6 & -24 & -120 & -720 & \dots \\ 0 & 1 & -3 & -3 & -12 & -60 & -360 & \dots \\ 0 & 0 & 1 & -4 & -4 & -20 & -120 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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$$R^4 = \begin{pmatrix} 1 & -1 & -12 & -78 & -504 & 36840 & -953280 & \cdots \\ -1 & 0 & -14 & -96 & -648 & 35640 & -923760 & \cdots \\ 19 & -21 & 13 & -39 & -312 & 17700 & -443880 & \cdots \\ -10 & 45 & -85 & 76 & -76 & 6000 & -141000 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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For $R^4$: $\frac{-78}{3!} = -13$, $\frac{-504}{4!} = -21$, $\frac{36840}{5!} = 307$, $\frac{-953280}{6!} = -1324$

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## Theorem

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$$R^5 = \begin{pmatrix} -2 & -11 & -42 & -156 & \dots \\ 1 & -13 & -52 & -216 & \dots \\ -40 & 36 & -74 & -183 & \dots \\ 55 & -165 & 261 & -335 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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**Lemma**

*For all* $m, n \geq 1$ *and for each* $0 \leq j \leq 2^m - 1$,

$$R_m^n(0, j) \equiv R^n(0, j) \mod 2^{2^m - 1}.$$

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$$R_1^4 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

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*For all* $m, n \geq 1$ *and for each* $0 \leq j \leq 2^m - 1$,

$$R^n_m(0, j) \equiv R^n(0, j) \mod 2^{2^m - 1}.$$

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$$R^4_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad R^4_2 = \begin{pmatrix} 1 & 7 & 4 & 2 \\ 7 & 0 & 2 & 0 \\ 3 & 7 & 1 & 5 \\ 6 & 1 & 3 & 4 \end{pmatrix}$$

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# The Top Left Entry

## Theorem

*For all* $n, m \in \mathbb{N}$,

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| 8 | 50 |
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$$-2 \equiv 22 \equiv 6 \mod 8 \qquad 267 \equiv 46203 \equiv 3 \mod 8$$

# Conclusion

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- Additional Results

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- Alternate Bases

# Acknowledgements

We would like to thank LSU for hosting the SMILE Program. Thank you NSF for funding the VIGRE program. Thank you to Dr. De Angelis for spending his summer with us. Thank you to Simon Pfeil for mentoring us.

# Works Cited


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