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JONATHAN BROOM, STEFAN HANNIE, AND SARAH SEGER

Abstract. T. Amdeberhan, V. De Angelis, and V.H. Moll study the arithmetical properties of the complementary Bell numbers using an infinite matrix P whose basis is the falling factorials. This paper constructs a similar matrix whose basis is the rising factorials and parallels many of the same results.

1. Introduction

The sequence of complementary Bell numbers, denoted $\widetilde{B}(n)$, quickly becomes unpredictable. It is known that $\widetilde{B}(2) = 0$, and H.S. Wilf conjectured that $\widetilde{B}(n) \neq 0$ for all $n > 2$. Although this conjecture has never been proven, it has been shown that $B(n) = 0$ for at most one $n > 2$. T. Amdeberhan, V. De Angelis, and V.H. Moll give an alternative proof of this fact using an infinite matrix constructed from the recurrence relation for the polynomials associated with the complementary Bell numbers. Their matrix, P, has a basis of falling factorials. In this paper, we will construct a similar matrix, R, that will perform the same function as P, but will have a basis of rising factorials. Then we will attempt to parallel the same results.

The sequence of complementary Bell numbers is related to the sequence of Bell numbers, and both are constructed from the Stirling numbers of the second kind. The *Stirling numbers of the second kind*, denoted $S(n,k)$, count the number of partitions of a finite set of size n into k blocks, where $n \in \mathbb{N}$ and $0 \leq k \leq n$. By convention, $S(0,0) = 1$ and $S(n,k) = 0$ for $k < 0$ or $k > n$.

The *Bell numbers*, denoted $B(n)$, are defined by

$$B(n) = \sum_{k=0}^{n} S(n,k)$$

and the *complementary Bell numbers*, denoted $\widetilde{B}(n)$, are defined by

$$\widetilde{B}(n) = \sum_{k=0}^{n} (-1)^k S(n,k), \tag{1.1}$$

for each $n \geq 0$. Hence $B(n)$ counts the total number of partitions of a set of size n, and $\widetilde{B}(n)$ counts the difference between the number of partitions with an even number of blocks and the number of those with an odd number of blocks.

T. Amdeberhan, V. De Angelis, and V.H. Moll have found a family of polynomials associated with the complementary Bell numbers. These polynomials, denoted

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λ_n are defined recursively by

$$\begin{aligned} \lambda_0(x) &= 1, \\ \lambda_{n+1}(x) &= x\lambda_n(x) - \lambda_n(x+1), \end{aligned} \tag{1.2}$$

for each $n \geq 0$. Note that λ_n is a monic polynomial of degree n. We investigate how these polynomials arose in Section 3.

By writing these polynomials in terms of falling factorials, T. Amdeberhan, V. De Angelis, and V.H. Moll translated the recursion equation for λ_n into the matrix equation $\mathbf{c}_{j+1} = P\mathbf{c}_j$, where $\mathbf{c}_{j+1}$ and $\mathbf{c}_j$ are the coefficent vectors for λ_{j+1} and λ_j, respectively. We will produce an analagous matrix equation $\mathbf{a}_{n+1} = R\mathbf{a}_n$ by writing the polynomials λ_n in terms of rising factorials.

Falling and rising factorials are variations on the more commonly used factorial, $x!$. The *falling factorial*, denoted $(x)_k$, is defined by

$$(x)_k = x(x-1)\cdots(x-k+1)$$

for all $k \in \mathbb{N}$. The *rising factorial*, denoted $x^{(k)}$, (also called the Pochhammer symbol) is analogously defined by

$$x^{(k)} = x(x+1)\cdots(x+k-1)$$

for all $k \in \mathbb{N}$. By convention, $(x)_0 = 1$ and $x^{(0)} = 1$. When n is a nonnegative integer, $(n)_k = \dfrac{n!}{(n-k)!}$ and $n^{(k)} = \dfrac{(n+k-1)!}{(n-1)!}$. However, we will interpret the falling and rising factorials as polynomials, which allows us to use them as bases for $\mathcal{P}$, the vector space of polynomials. Note that $(x)_k$ and $x^{(k)}$ both have degree k, and we can translate between the two definitions just by reversing the ordering of the factors. Thus $(x)_k = (x-k+1)^{(k)}$ and $x^{(k)} = (x+k-1)_k$. This relationship explains the motivation for replacing the falling factorials with the rising factorials in our study of the complementary Bell numbers.

2. The Basis of Rising Factorials

In this section we will prove the earlier assertion that the rising factorials form a basis for $\mathcal{P}$, the vector space of polynomials. However, we will first need an identity about the rising factorials. This identity is presented without proof in Lemma 4.2 of [1], so we will prove it here.

Lemma 2.1. *For all $k \geq 0$, the rising factorials satisfy the identity*

$$x \cdot x^{(k)} = x^{(k+1)} - kx^{(k)}.$$

Proof. Let $k \geq 0$. Then

$$\begin{aligned} x \cdot x^{(k)} &= (x+k-k)\,x^{(k)} \\ &= (x+k)\,x^{(k)} - kx^{(k)} \\ &= x(x+1)\cdots(x+k-1)(x+k) - kx^{(k)} \\ &= x^{(k+1)} - kx^{(k)}. \end{aligned}$$

□

Equipped with this identity, will now be able to prove that the rising factorials form a basis for $\mathcal{P}$. The usual basis for $\mathcal{P}$ is $\left\{1, x, x^2, \ldots\right\}$. The following proposition will show not only that each of these basis vectors can be written as a linear combination of rising factorials, but that the coefficients of the linear combination are the Stirling numbers of the second kind. A similar result holds when using the falling factorials as a basis. Since the complementary Bell numbers are constructed from the Stirling numbers, these results show the motivation for using the falling or rising factorials instead of the usual basis.

Proposition 2.2. *For all $n \geq 0$,*

$$x^n = \sum_{k=0}^{n} (-1)^{n+k} S(n,k) x^{(k)}.$$

Proof. The proof will proceed by induction on n. The $n = 0$ case holds, since $x^0 = 1 = (-1)^0 S(0,0) x^{(0)}$. Now assume that the proposition holds for some $n \geq 0$. Then

$$x^{n+1} = x \cdot x^n = x \sum_{k=0}^{n} (-1)^{n+k} S(n,k) x^{(k)} = \sum_{k=0}^{n} (-1)^{n+k} S(n,k) x \cdot x^{(k)}.$$

Applying Lemma 2.1, we find that

$$\begin{aligned} x^{n+1} &= \sum_{k=0}^{n} (-1)^{n+k} S(n,k) \left(x^{(k+1)} - k x^{(k)} \right) \\ &= \sum_{k=0}^{n} (-1)^{n+k} S(n,k) x^{(k+1)} - \sum_{k=0}^{n} (-1)^{n+k} k S(n,k) x^{(k)} \\ &= \sum_{k=1}^{n+1} (-1)^{n+k-1} S(n,k-1) x^{(k)} - \sum_{k=0}^{n} (-1)^{n+k} k S(n,k) x^{(k)}. \end{aligned}$$

Then since $S(n,-1) = 0 = S(n,n+1)$,

$$\begin{aligned} x^{n+1} &= \sum_{k=0}^{n+1} (-1)^{n+k-1} S(n,k-1) x^{(k)} - \sum_{k=0}^{n+1} (-1)^{n+k} k S(n,k) x^{(k)} \\ &= \sum_{k=0}^{n+1} (-1)^{n+k} \left(-S(n,k-1) - k S(n,k) \right) x^{(k)} \\ &= \sum_{k=0}^{n+1} (-1)^{n+k+1} \left(S(n,k-1) + k S(n,k) \right) x^{(k)} \\ &= \sum_{k=0}^{n+1} (-1)^{(n+1)+k} S(n+1,k) x^{(k)}, \end{aligned}$$

by the recurrence relation for $S(n,k)$, Equation 3.1. Therefore the proposition holds for all $n \geq 0$. $\square$

Finally, we are prepared to prove the main result of this section.

Theorem 2.3. *For each $n \geq 0$, the set of rising factorials $\left\{x^{(k)} : 0 \leq k \leq n\right\}$ is a basis for $\mathcal{P}_n$, the vector space of polynomials of degree less than or equal to n.*

Proof. Let $n \geq 0$. The usual basis for $\mathcal{P}_n$ is $\{1, x, \ldots, x^n\}$. Let $0 \leq j \leq n$. Then by Proposition 2.2,

$$x^j = \sum_{k=0}^{j} (-1)^{j+k} S(j,k) x^{(k)} = \sum_{k=0}^{n} (-1)^{j+k} S(j,k) x^{(k)},$$

since $S(j,k) = 0$ when $k > j$. Hence x^j can be written as a linear combination of $\left\{x^{(k)} : 0 \leq k \leq n\right\}$ for each $0 \leq j \leq n$. Then since $\left\{x^{(k)} : 0 \leq k \leq n\right\}$ spans a basis of $\mathcal{P}_n$, it spans the entire vector space. Thus since it is a spanning set of $\mathcal{P}_n$ with the same number of vectors, $n+1$, as a basis of $\mathcal{P}_n$, it must be a basis for $\mathcal{P}_n$. □

3. The Polynomials Associated with the Complementary Bell Numbers

The polynomials λ_n arise from the attempt to establish a recurrence relation for $\widetilde{B}(n)$. By the definition of $\widetilde{B}(n)$, Equation 1.1,

$$\widetilde{B}(n+1) = \sum_{k=0}^{n+1} (-1)^k S(n+1,k).$$

It can be proven combinatorically that $S(n,k)$ obeys the recurrence relation

$$S(n+1,k) = S(n,k-1) + kS(n,k). \tag{3.1}$$

. Thus applying this relation and then by manipulating the summation, we find that

$$\widetilde{B}(n+1) = \sum_{k=0}^{n} (-1)^k \left(k-1\right) S(n,k).$$

This is not a recurrence relation, but it does lead to the conjecture that $\widetilde{B}(n+j)$ can be written as the alternating summation over k of $S(n,k)$ and a polynomial of degree j.

The following results are given in Lemma 7.1 and Corollary 7.2 of [1], however the proofs are left to the reader, so we will prove them here.

Proposition 3.1. *For all $n \geq 0$ and for all $j \geq 0$,*

$$\widetilde{B}(n+j) = \sum_{k=0}^{n} (-1)^k \lambda_j(k) S(n,k),$$

where λ_j denotes the polynomials defined in Section 1.

Proof. The proof will proceed by induction on j. The $j = 0$ case holds for all $n \geq 0$ by the definition of $\widetilde{B}(n)$ in Equation 1.1. Now assume that the proposition holds for all $n \geq 0$ for some $j \geq 0$. Let $n \geq 0$. Then

$$\widetilde{B}(n+(j+1)) = \widetilde{B}((n+1)+j) = \sum_{k=0}^{n+1} (-1)^k \lambda_j(k) S(n+1,k).$$

Applying the recurrence relation for $S(n,k)$, Equation 3.1, we find that

$$\begin{aligned}\widetilde{B}(n+(j+1)) &= \sum_{k=0}^{n+1}(-1)^k\lambda_j(k)\left(S(n,k-1)+kS(n,k)\right)\\ &= \sum_{k=0}^{n+1}(-1)^k\lambda_j(k)S(n,k-1)+\sum_{k=0}^{n+1}(-1)^k\lambda_j(k)kS(n,k)\\ &= \sum_{k=-1}^{n}(-1)^{k+1}\lambda_j(k+1)S(n,k)+\sum_{k=0}^{n+1}(-1)^k k\lambda_j(k)S(n,k)\\ &= \sum_{k=0}^{n}(-1)^k\left(-\lambda_j(k+1)+k\lambda_j(k)\right)S(n,k)\end{aligned}$$

since $S(n,-1)=0=S(n,n+1)$. Applying the recursion for λ_n, Equation 1.2,

$$\widetilde{B}(n+(j+1)) = \sum_{k=0}^{n}(-1)^k\lambda_{j+1}(k)S(n,k).$$

Therefore by induction the proposition holds for all $n \geq 0$ for all $j \geq 0$. □

Corollary 3.2. *For all $j \geq 0$, $\widetilde{B}(j)=\lambda_j(0)$.*

Proof. By Proposition 3.1,

$$\widetilde{B}(j)=\widetilde{B}(0+j)=\sum_{k=0}^{0}(-1)^k\lambda_j(k)S(0,k)=(-1)^0\lambda_j(0)S(0,0)=\lambda_j(0)$$

for all $j \geq 0$. □

So we do not have a recursion for $\widetilde{B}(n)$, but we do have a recursion for λ_n, Equation 1.2, and since these polynomials are closely related to the complementary Bell numbers, this recursion might have importance for the complementary Bell numbers. We can translate this recursion into a matrix equation by writing these polynomials in terms of rising factorials.

Let $n \geq 0$. Then λ_n is a polynomial of degree n, thus $\lambda_n \in \mathcal{P}_n$. By Theorem 2.3 the set of rising factorials $\left\{x^{(k)} : 0 \leq k \leq n\right\}$ is a basis for $\mathcal{P}_n$, hence there exist unique scalars $a_n(0),\ldots,a_n(n)$ such that

$$\lambda_n(x)=\sum_{k=0}^{n}a_n(k)x^{(k)}. \tag{3.2}$$

Then let $\mathbf{a}_n = (a_n(0),\ldots,a_n(n),a_n(n+1),\ldots)$ denote the infinite vector whose first $n+1$ components are the above scalars and $a_n(k)=0$ if $k>n$. We will also define $a_n(k)=0$ if $k<0$, which will not appear in the vector, but will be useful in later proofs.

Now we wish to find an infinite matrix R such that $\mathbf{a}_{n+1}=R\mathbf{a}_n$ for all $n \geq 0$. We have found two different methods of constructing this matrix.

4. Construction of the Matrix R (The First Method)

For both methods, we will construct R by translating the recursion for λ_n, Equation 1.2, into a matrix equation. For this method, we will rewrite the recursion as

$$\lambda_{n+1}(x-1) = (x-1)\,\lambda_n(x-1) - \lambda_n(x).$$

By Equation 3.2, this is equivalent to

$$\sum_{k=0}^{n+1} a_{n+1}(k)\,(x-1)^{(k)} = \sum_{k=0}^{n} a_n(k)\,(x-1)\,(x-1)^{(k)} - \sum_{k=0}^{n} a_n(k) x^{(k)}. \tag{4.1}$$

In order to write both sides of this equation in terms of $x^{(k)}$, we will need another identity about the rising factorials. This identity is also presented in Lemma 4.2 of [1], but again without proof, so we will prove it here.

Lemma 4.1. *For all $k \in \mathbb{N}$, the rising factorials satisfy the identity*

$$(x-1)^{(k)} = x^{(k)} - kx^{(k-1)}.$$

Proof. Let $k \in \mathbb{N}$. Then

$$\begin{aligned}
(x-1)^{(k)} &= (x-1)\,x\,(x+1)\cdots(x+k-2) \\
&= (x+k-1-k)\,x\,(x+1)\cdots(x+k-2) \\
&= x\,(x+1)\cdots(x+k-2)\,(x+k-1) - kx\,(x+1)\cdots(x+k-2) \\
&= x^{(k)} - kx^{(k-1)}.
\end{aligned}$$

□

Now we can manipulate Equation 4.1 so that both sides are written in terms of $x^{(k)}$, and find an equation relating the coefficients, which brings us to the following Lemma.

Lemma 4.2. *For all $n \geq 0$ and for all $0 \leq k \leq n+1$,*

$$a_{n+1}(k) - (k+1)\,a_{n+1}(k+1) = a_n(k-1) - 2\,(k+1)\,a_n(k) + (k+1)^2\,a_n(k+1).$$

Proof. Let $n \geq 0$. By Lemma 4.1, the left side of Equation 4.1

$$\begin{aligned}
\sum_{k=0}^{n+1} a_{n+1}(k)\,(x-1)^{(k)} &= \sum_{k=0}^{n+1} a_{n+1}(k)\left(x^{(k)} - kx^{(k-1)}\right) \\
&= \sum_{k=0}^{n+1} a_{n+1}(k)x^{(k)} - \sum_{k=0}^{n+1} a_{n+1}(k)kx^{(k-1)} \\
&= \sum_{k=0}^{n+1} a_{n+1}(k)x^{(k)} - \sum_{k=-1}^{n} (k+1)\,a_{n+1}(k+1)x^{(k)} \\
&= \sum_{k=0}^{n+1} a_{n+1}(k)x^{(k)} - \sum_{k=0}^{n+1} (k+1)\,a_{n+1}(k+1)x^{(k)} \\
&= \sum_{k=0}^{n+1} \left(a_{n+1}(k) - (k+1)\,a_{n+1}(k+1)\right)x^{(k)},
\end{aligned} \tag{4.2}$$

since $a_{n+1}(n+2)=0$. To similarly manipulate the right side of Equation 4.1,

$$\sum_{k=0}^{n} a_n(k)\,(x-1)\,(x-1)^{(k)} - \sum_{k=0}^{n} a_n(k)x^{(k)},$$

we first note that by Lemma 2.1 and Lemma 4.1,

$$\begin{aligned}(x-1)\,(x-1)^{(k)} &= (x-1)^{(k+1)} - k\,(x-1)^{(k)}\\ &= \left(x^{(k+1)} - (k+1)\,x^{(k)}\right) - k\left(x^{(k)} - kx^{(k-1)}\right)\\ &= x^{(k+1)} - (2k+1)\,x^{(k)} + k^2x^{(k-1)}\end{aligned}$$

for all $0 \le k \le n$. Hence the right side of Equation 4.1 equals

$$\begin{aligned}&= \sum_{k=0}^{n} a_n(k)\left(x^{(k+1)} - (2k+1)\,x^{(k)} + k^2x^{(k-1)}\right) - \sum_{k=0}^{n} a_n(k)x^{(k)}\\ &= \sum_{k=0}^{n} a_n(k)\left(x^{(k+1)} - 2\,(k+1)\,x^{(k)} + k^2x^{(k-1)}\right)\\ &= \sum_{k=0}^{n} a_n(k)x^{(k+1)} - 2\sum_{k=0}^{n}(k+1)\,a_n(k)x^{(k)} + \sum_{k=0}^{n} k^2a_n(k)x^{(k-1)}\\ &= \sum_{k=1}^{n+1} a_n(k-1)x^{(k)} - 2\sum_{k=0}^{n}(k+1)\,a_n(k)x^{(k)} + \sum_{k=-1}^{n-1}(k+1)^2\,a_n(k+1)x^{(k)}\\ &= \sum_{k=0}^{n+1} a_n(k-1)x^{(k)} - 2\sum_{k=0}^{n}(k+1)\,a_n(k)x^{(k)} + \sum_{k=0}^{n+1}(k+1)^2\,a_n(k+1)x^{(k)}\\ &= \sum_{k=0}^{n+1}\left(a_n(k-1) - 2\,(k+1)\,a_n(k) + (k+1)^2\,a_n(k+1)\right), \qquad (4.3)\end{aligned}$$

since $a_n(-1)=a_n(n+1)=a_n(n+2)=0$. Then combining Equation 4.1 with Equations 4.2 and 4.3, we get

$$\begin{aligned}&\sum_{k=0}^{n+1}\left(a_{n+1}(k) - (k+1)\,a_{n+1}(k+1)\right)x^{(k)}\\ &\qquad = \sum_{k=0}^{n+1}\left(a_n(k-1) - 2\,(k+1)\,a_n(k) + (k+1)^2\,a_n(k+1)\right)\end{aligned}$$

Therefore since $\left\{x^{(k)} : 0 \le k \le n+1\right\}$ is a basis for $\mathcal{P}_{n+1}$ by Theorem 2.3,

$$a_{n+1}(k) - (k+1)\,a_{n+1}(k+1) = a_n(k-1) - 2\,(k+1)\,a_n(k) + (k+1)^2\,a_n(k+1)$$

for all $0 \le k \le n+1$. $\square$

This result yields a system of $n+1$ equations for each $n \ge 0$ which we will translate into a matrix equation. Since we are working in a polynomial vector space, it is convenient to index the rows and columns of matrices starting with 0 rather than 1. We will also notate the (i,j)th entry of a matrix M by $M(i,j)$.

Proposition 4.3. *Let A, B be infinite matrices whose entries are defined by*

$$A(i,j) = \begin{cases} 1 & \text{if } j = i \\ -(i+1) & \text{if } j = i+1 \\ 0 & \text{if } j < i \\ & \text{or } j > i+1 \end{cases} \quad \text{and } B(i,j) = \begin{cases} 1 & \text{if } j = i-1 \\ -2(i+1) & \text{if } j = i \\ (i+1)^2 & \text{if } j = i+1 \\ 0 & \text{if } |i-j| > 1 \end{cases}$$

So

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & \cdots \\ 0 & 1 & -2 & 0 & \cdots \\ 0 & 0 & 1 & -3 & \ddots \\ 0 & 0 & 0 & 1 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix} \quad \text{and } B = \begin{pmatrix} -2 & 1 & 0 & 0 & \cdots \\ 1 & -4 & 4 & 0 & \cdots \\ 0 & 1 & -6 & 9 & \ddots \\ 0 & 0 & 1 & -8 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

Then $A\mathbf{a}_{n+1} = B\mathbf{a}_n$ for all $n \geq 0$.

Proof. Let $n \geq 0$. Denote the kth entry of the column vector $A\mathbf{a}_{n+1}$ by $(A\mathbf{a}_{n+1})(k)$, and similarly denote the kth entry of $B\mathbf{a}_n$. If $0 \leq k \leq n+1$, then

$$\begin{aligned} (A\mathbf{a}_{n+1})(k) &= \sum_{j=0}^{\infty} A(k,j) a_{n+1}(j) \\ &= A(k,k)a_{n+1}(k) + A(k,k+1)a_{n+1}(k+1) \\ &= a_{n+1}(k) - (k+1)\, a_{n+1}(k+1) \\ &= a_n(k-1) - 2(k+1)\, a_n(k) + (k+1)^2\, a_n(k+1) \\ &= B(k,k-1)a_n(k-1) + B(k,k)a_n(k) + B(k,k+1)a_n(k+1) \\ &= \sum_{j=0}^{\infty} B(k,j) a_n(j) \\ &= (B\mathbf{a}_n)(k), \end{aligned}$$

by Lemma 4.2. If $k > n+1$, then $(A\mathbf{a}_{n+1})(k) = 0 = (B\mathbf{a}_n)(k)$. Therefore $A\mathbf{a}_{n+1} = B\mathbf{a}_n$. □

Now we have the matrix equation $A\mathbf{a}_{n+1} = B\mathbf{a}_n$. In order to translate this into a recursion equation for $\mathbf{a}_n$ with only one matrix, we will need to find the inverse of A. We find that A^{-1} has a very pleasing pattern of entries. Going up the columns we get falling factorials, and going along the rows we get risng factorials.

Proposition 4.4. *Let C be an infinite matrix whose entries are defined by*

$$C(i,j) = \begin{cases} 0 & \text{if } j < i \\ \dfrac{j!}{i!} & \text{if } j \geq i \end{cases}, \text{ so } C = \begin{pmatrix} 1 & 1 & 2 & 6 & \cdots \\ 0 & 1 & 2 & 6 & \cdots \\ 0 & 0 & 1 & 3 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

Then $C = A^{-1}$.

Proof. It suffices to show that $AC = CA = I_\infty$, where I_∞ is the infinite dimensional identity matrix. The (i, j)th entry of AC is given by

$$\begin{aligned} AC(i,j) &= \sum_{k=0}^{\infty} A(i,k)C(k,j) \\ &= A(i,i)C(i,j) + A(i,i+1)C(i+1,j) \\ &= C(i,j) - (i+1)\,C(i+1,j) \end{aligned}$$

by the definition of A in Proposition 4.3.

If $j < i$, then $C(i,j) - (i+1)\,C(i+1,j) = 0$. If $j = i$, then

$$C(i,i) - (i+1)\,C(i+1,i) = \frac{i!}{i!} - 0 = 1.$$

If $i < j$, then

$$C(i,j) - (i+1)\,C(i+1,j) = \frac{j!}{i!} - (i+1)\,\frac{j!}{(i+1)!} = \frac{j!}{i!} - \frac{j!}{i!} = 0.$$

Therefore $AC(i,j) = 1$ if $i = j$ and $AC(i,j) = 0$ if $i \neq j$, hence $AC = I_\infty$.

The entries of CA are given by

$$\begin{aligned} CA(i,j) &= \sum_{k=0}^{\infty} C(i,k)A(k,j) \\ &= C(i,j-1)A(j-1,j) + C(i,j)A(j,j) \\ &= -jC(i,j-1) + C(i,j), \end{aligned}$$

again by definition of A. If $j < i$, then $-jC(i,j-1) + C(i,j) = 0$. If $j = i$, then

$$-iC(i,i-1) + C(i,i) = 0 + \frac{i!}{i!} = 1.$$

If $j > i$, then

$$-jC(i,j-1) + C(i,j) = -j\frac{(j-1)!}{i!} + \frac{j!}{i!} = -\frac{j!}{i!} + \frac{j!}{i!} = 0.$$

Therefore $CA(i,j) = 1$ if $i = j$ and $CA(i,j) = 0$ if $i \neq j$, hence $CA = I_\infty$. $\square$

Now that we've found A^{-1} we need only multiply by B to find R.

Proposition 4.5. *Let R be the infinite matrix whose entries are defined*

$$R(i,j) = \begin{cases} 0 & \text{if } j < i-1 \\ 1 & \text{if } j = i-1 \\ -(i+1) & \text{if } j = i \\ -\frac{j!}{i!} & \text{if } j > i. \end{cases} \quad \text{so } R = \begin{pmatrix} -1 & -1 & -2 & -6 & \cdots \\ 1 & -2 & -2 & -6 & \cdots \\ 0 & 1 & -3 & -3 & \cdots \\ 0 & 0 & 1 & -4 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Then $R = A^{-1}B$.

Proof. The (i, j)th entry of $A^{-1}B$ is given by

$$\begin{aligned} A^{-1}B(i,j) &= \sum_{k=0}^{\infty} A^{-1}(i,k)B(k,j) \\ &= A^{-1}(i,j-1)B(j-1,j) + A^{-1}(i,j)B(j,j) + A^{-1}(i,j+1)B(j+1,j) \\ &= j^2 A^{-1}(i,j-1) - 2\,(j+1)\,A^{-1}(i,j) + A^{-1}(i,j+1). \end{aligned}$$

If $j < i - 1$, then $A^{-1}B(i, j) = 0$. If $j = i - 1$, then

$$\begin{aligned} A^{-1}B(i, i-1) &= j^2 A^{-1}(i, i-2) - 2iA^{-1}(i, i-1) + A^{-1}(i, i) \\ &= 0 + 0 + \frac{i!}{i!} \\ &= 1. \end{aligned}$$

If $j = i$, then

$$\begin{aligned} A^{-1}B(i, i) &= i^2 A^{-1}(i, i-1) - 2\,(i+1)\,A^{-1}(i, i) + A^{-1}(i, i+1) \\ &= 0 - 2\,(i+1)\,\frac{i!}{i!} + \frac{(i+1)!}{i!} \\ &= -2\,(i+1) + (i+1) \\ &= -\,(i+1)\,. \end{aligned}$$

If $j > i$, then

$$\begin{aligned} A^{-1}B(i, j) &= j^2 A^{-1}(i, j-1) - 2\,(j+1)\,A^{-1}(i, j) + A^{-1}(i, j+1) \\ &= j^2 \frac{(j-1)!}{i!} - 2\,(j+1)\,\frac{j!}{i!} + \frac{(j+1)!}{i!} \\ &= j\frac{j!}{i!} - 2\,(j+1)\,\frac{j!}{i!} + (j+1)\,\frac{j!}{i!} \\ &= \frac{j!}{i!}\,(j - (j+1)) \\ &= -\frac{j!}{i!} \end{aligned}$$

Therefore the entries of R match those of $A^{-1}B$. □

Now we have defined R and shown that it satisfies $\mathbf{a}_{n+1} = R\mathbf{a}_n$ for all $n \geq 0$.

5. Construction of the Matrix R (The Second Method)

For this method of constructing R, we will use the original recursion for λ_n,

$$\lambda_{n+1}(x) = x\lambda_n(x) - \lambda_n\,(x+1)\,.$$

By Equation 3.2, this is equivalent to

$$\sum_{k=0}^{n+1} a_{n+1}(k)x^{(k)} = x\sum_{k=0}^{n} a_n(k)x^{(k)} - \sum_{k=0}^{n} a_n(k)\,(x+1)^{(k)}\,. \tag{5.1}$$

In order to rewrite the right side of the equation in terms of $x^{(k)}$, we will need yet another identity about the rising factorials.

Lemma 5.1. *For all $k \geq 0$, the rising factorials satsify the identity*

$$(x+1)^{(k)} = \sum_{j=0}^{k} \frac{k!}{j!} x^{(j)}.$$

Proof. The proof will proceed by induction on k. The n=0 case holds, since $(x+1)^{(0)} = 1 = \frac{0!}{0!}x^{(0)}$. Now assume the lemma is true for some $k \geq 0$. Then

$$\begin{aligned}
(x+1)^{(k+1)} &= (x+1)\cdots(x+k)(x+k+1) \\
&= (x+1)^{(k)}(x+k+1) \\
&= (x+k+1)\sum_{j=0}^{k}\frac{k!}{j!}x^{(j)} \\
&= \sum_{j=0}^{k}\frac{k!}{j!}x\cdot x^{(j)} + \sum_{j=0}^{k}(k+1)\frac{k!}{j!}x^{(j)}. \qquad (5.2)
\end{aligned}$$

By Lemma 2.1,

$$\begin{aligned}
\sum_{j=0}^{k}\frac{k!}{j!}x\cdot x^{(j)} &= \sum_{j=0}^{k}\frac{k!}{j!}\left(x^{(j+1)} - jx^{(j)}\right) \\
&= \sum_{j=0}^{k}\frac{k!}{j!}x^{(j+1)} - \sum_{j=0}^{k}\frac{k!}{j!}jx^{(j)} \\
&= \sum_{j=1}^{k+1}\frac{k!}{(j-1)!}x^{(j)} - \sum_{j=1}^{k}\frac{k!}{j!}jx^{(j)} \\
&= x^{(k+1)} + \sum_{j=1}^{k}\frac{k!}{(j-1)!}x^{(j)} - \sum_{j=1}^{k}\frac{k!}{(j-1)!}x^{(j)} \\
&= x^{(k+1)}.
\end{aligned}$$

Then continuing from Equation 5.2,

$$\begin{aligned}
(x+1)^{(k+1)} &= x^{(k+1)} + \sum_{j=0}^{k}(k+1)\frac{k!}{j!}x^{(j)} \\
&= x^{(k+1)} + \sum_{j=0}^{k}\frac{(k+1)!}{j!}x^{(j)} \\
&= \sum_{j=0}^{k+1}\frac{(k+1)!}{j!}x^{(j)}.
\end{aligned}$$

Therefore by induction the lemma is true for all $k \in \mathbb{N}$. □

With this identity, we can now manipulate Equation 5.1 and obtain a recursion equation for the coefficients $a_n(k)$.

Lemma 5.2. *For all $n \geq 0$ and for each $k = 0, \ldots, n+1$,*

$$a_{n+1}(k) = a_n(k-1) - (k+1)\,a_n(k) - \sum_{j=k+1}^{n}\frac{j!}{k!}a_n(j)$$

.

Proof. Let $n \geq 0$. First we must manipulate the right side of Equation 5.1 so that it is in terms of $x^{(k)}$. Applying Lemma 2.1 to the first term of the right side, we have

$$
\begin{aligned}
x \sum_{k=0}^{n} a_n(k) x^{(k)} &= \sum_{k=0}^{n} a_n(k) x \cdot x^{(k)} \\
&= \sum_{k=0}^{n} a_n(k) \left(x^{(k+1)} - k x^{(k)} \right) \\
&= \sum_{k=0}^{n} a_n(k) x^{(k+1)} - \sum_{k=0}^{n} k a_n(k) x^{(k)} \\
&= \sum_{k=1}^{n+1} a_n(k-1) x^{(k)} - \sum_{k=0}^{n} k a_n(k) x^{(k)} \\
&= \sum_{k=0}^{n+1} a_n(k-1) x^{(k)} - \sum_{k=0}^{n+1} k a_n(k) x^{(k)} \\
&= \sum_{k=0}^{n+1} \left(a_n(k-1) - k a_n(k) \right) x^{(k)}. \qquad (5.3)
\end{aligned}
$$

Applying Lemma 5.1 to the second term of the right side of Equation 5.1, we get

$$
\begin{aligned}
\sum_{k=0}^{n} a_n(k) \left(x+1 \right)^{(k)} &= \sum_{k=0}^{n} a_n(k) \sum_{j=0}^{k} \frac{k!}{j!} x^{(j)} \\
&= \sum_{k=0}^{n} a_n(k) \left(\frac{k!}{0!} x^{(0)} + \frac{k!}{1!} x^{(1)} + \cdots + \frac{k!}{k!} x^{(k)} \right) \\
&= \quad a_n(0) \left(\frac{0!}{0!} x^{(0)} \right) \\
&\quad + a_n(1) \left(\frac{1!}{0!} x^{(0)} + \frac{1!}{1!} x^{(1)} \right) \\
&\quad \vdots \\
&\quad + a_n(n) \left(\frac{n!}{0!} x^{(0)} + \frac{n!}{1!} x^{(1)} + \cdots + \frac{n!}{n!} x^{(n)} \right) \\
&= \sum_{j=0}^{n} \frac{j!}{0!} a_n(j) x^{(0)} + \sum_{j=1}^{n} \frac{j!}{1!} a_n(j) x^{(1)} + \cdots + \sum_{j=n}^{n} \frac{j!}{n!} a_n(j) x^{(n)} \\
&= \sum_{k=0}^{n} \sum_{j=k}^{n} \frac{j!}{k!} a_n(j) x^{(k)} \\
&= \sum_{k=0}^{n+1} \sum_{j=k}^{n} \frac{j!}{k!} a_n(j) x^{(k)}, \qquad (5.4)
\end{aligned}
$$

where $\sum_{j=n+1}^{n} \frac{j!}{k!} a_n(j) x^{(k)}$ is understood to be 0. Then combining Equations 5.1, 5.3, and 5.4, we have

$$\begin{aligned}
\sum_{k=0}^{n+1} a_{n+1}(k) x^{(k)} &= \sum_{k=0}^{n+1} \left(a_n(k-1) - k a_n(k)\right) x^{(k)} - \sum_{k=0}^{n+1} \sum_{j=k}^{n} \frac{j!}{k!} a_n(j) x^{(k)} \\
&= \sum_{k=0}^{n+1} \left(a_n(k-1) - k a_n(k) - \sum_{j=k}^{n} \frac{j!}{k!} a_n(j) \right) x^{(k)} \\
&= \sum_{k=0}^{n+1} \left(a_n(k-1) - k a_n(k) - \frac{k!}{k!} a_n(k) - \sum_{j=k+1}^{n} \frac{j!}{k!} a_n(j) \right) x^{(k)} \\
&= \sum_{k=0}^{n+1} \left(a_n(k-1) - (k+1)\, a_n(k) - \sum_{j=k+1}^{n} \frac{j!}{k!} a_n(j) \right) x^{(k)}.
\end{aligned}$$

Therefore since the rising factorials form a basis for $\mathcal{P}_n$ by Theorem 2.3,

$$a_{n+1}(k) = a_n(k-1) - (k+1)\, a_n(k) - \sum_{j=k+1}^{n} \frac{j!}{k!} a_n(j)$$

for each $k = 0, \ldots, n+1$. □

This gives us a system of equations relating the coefficients. Now we are ready to prove the main theorem of this section.

Theorem 5.3. *The matrix equation* $\mathbf{a}_{n+1} = R\mathbf{a}_n$*, where* R *is defined as in Proposition 4.5, holds for all* $n \geq 0$*.*

Proof. Let $n \geq 0$. Let $(R\mathbf{a}_n)(k)$ denote the kth entry of the column vector $R\mathbf{a}_n$. By the definition of R, the first nonzero entry on any ith row is $R(i, i-1)$, so

$$\begin{aligned}
(R\mathbf{a}_n)(k) &= \sum_{j=0}^{\infty} R(k,j) a_n(j) \\
&= R(k,k-1) a_n(k-1) + R(k,k) a_n(k) + \sum_{j=k+1}^{\infty} R(k,j) a_n(j) \\
&= a_n(k-1) - (k+1)\, a_n(k) - \sum_{j=k+1}^{\infty} \frac{j!}{k!} a_n(j). \\
&= a_n(k-1) - (k+1)\, a_n(k) - \sum_{j=k+1}^{n} \frac{j!}{k!} a_n(j).
\end{aligned}$$

since $a_n(j) = 0$ if $j > n$. If $0 \leq k \leq n+1$, then $(R\mathbf{a}_n)(k) = a_{n+1}(k)$ by Lemma 5.2. If $k > n+1$, then $j > n$ for all $j = k-1, k, k+1, \ldots$, so $a_n(j) = 0$ for all $j = k-1, k, k+1, \ldots$, hence $(R\mathbf{a}_n)(k) = 0 = a_{n+1}(k)$. Thus every row of the matrix equation holds true. Therefore $\mathbf{a}_{n+1} = R\mathbf{a}_n$. □

Therefore both constructions yield the same matrix R, and R satisfies the matrix equation $\mathbf{a}_{n+1} = R\mathbf{a}_n$, which is a recursion for $\mathbf{a}_n$.

6. Results

Now that we have constructed R we will try to parallel the results that T. Amdeberhan, V. De Angelis, and V.H. Moll found with their matrix P. First we will need to be able to take powers of R.

Since R is an infinite matrix, the entries of R^2 are given by infinite sums, so it is not immediately clear whether R^2 is defined. However, as can be seen from the definition of R in Proposition 4.5, each column of R has only finitely many nonzero entries. Thus each infinite sum defining the entries of R^2 has only finitely many nonzero entries. This leads us to the following proposition.

Proposition 6.1. *For each $n \in \mathbb{N}$, the nth power of R is defined and $R^n(i,j) = 0$ if $j < i - n$.*

Proof. The $n = 1$ case holds by definition of R. Assume that the proposition holds for some $n \in \mathbb{N}$. The $(i,j)th$ entry of R^{n+1} is given by

$$R^{n+1}(i,j) = \sum_{k=0}^{\infty} R^n(i,k)R(k,j) = \sum_{k=0}^{j+1} R^n(i,k)R(k,j)$$

since $j < k - 1$ for all $k \geq j + 2$. Then since this is only a finite sum, every entry of R^{n+1} is defined. If $j < i - n - 1$, then $k < i - n$ for all $k \leq j + 1$, hence by the induction hypothesis $R^n(k,j) = 0$ for all $k \leq j + 1$. Hence $R^{n+1}(i,j) = 0$ when $j < i - (n + 1)$. Therefore by induction, the lemma holds for all $n \in \mathbb{N}$. □

Now that we know R^n is defined, we can explore other aspects of its structure. For instance, the first nonzero diagonal consits entirely of 1s.

Proposition 6.2. *For all $n \in \mathbb{N}$, $R^n(i, i - n) = 1$.*

Proof. The $n = 1$ case holds by definition of R. Assume that the proposition holds for some $n \in \mathbb{N}$. The $(i, i - n - 1)th$ entry of R^{n+1} is given by

$$R^{n+1}(i, i-n-1) = \sum_{k=0}^{\infty} R^n(i,k)R(k, i-n-1) = \sum_{k=0}^{i-n} R^n(i,k)R(k, i-n-1)$$

since $i - n - 1 < k - 1$ for all $k \geq i - n + 1$. By Proposition 6.1, $R^n(i,k) = 0$ for all $k < i - n$, hence

$$R^{n+1}(i, i-n-1) = R^n(i, i-n)R(i-n, i-n-1) = 1$$

by the induction hypothesis and the definition of R. □

We have also found experimentally that $R^n(i, i - k)$ is a polynomial of degree $n - k$ for all $0 \leq k \leq n$, but this is harder to prove. Another interesting aspect of the structure of R^n is that on the main diagonal and every diagonal above, $R^n(i,j)$ is divisible by $\frac{j!}{i!}$. Of course on the main diagonal, $i = j$ so $\frac{j!}{i!} = 1$, so this case is not as interesting.

Proposition 6.3. *For all $n \geq 1$ and for all $j > i$, the $(i,j)th$ entry of R^n is divisible by $\frac{j!}{i!}$.*

Proof. The proof will proceed by induction on n. The $n=1$ case holds by definition of R. Now suppose that the proposition holds for some $n \geq 1$. Then for all $k \geq i$, there exists an integer $q_k \geq 0$ such that $R^n(i,k) = \frac{k!}{i!} q_k$. Let $j \geq i$. Then the (i,j)th entry of R^{n+1} is given by

$$\begin{aligned}
R^{n+1}(i,j) &= \sum_{k=0}^{\infty} R^n(i,k)R(k,j) = \sum_{k=0}^{j+1} R^n(i,k)R(k,j) \\
&= -\sum_{k=0}^{j-1} \frac{j!}{k!} R^n(i,k) - (j+1)\,R^n(i,j) + R^n(i,j+1) \\
&= -\sum_{k=0}^{i-1} \frac{j!}{k!} R^n(i,k) - \sum_{k=i}^{j-1} \frac{j!}{k!} R^n(i,k) - (j+1)\,R^n(i,j) + R^n(i,j+1) \\
&= -\sum_{k=0}^{i-1} \frac{j!}{k!}\frac{i!}{i!} R^n(i,k) - \sum_{k=i}^{j-1} \frac{j!}{k!}\frac{k!}{i!} q_k - (j+1)\,\frac{j!}{i!} q_j + \frac{(j+1)!}{i!} q_{j+1} \\
&= -\sum_{k=0}^{i-1} \frac{j!}{i!}\frac{i!}{k!} R^n(i,k) - \sum_{k=i}^{j-1} \frac{j!}{i!} q_k - (j+1)\,\frac{j!}{i!} q_j + (j+1)\,\frac{j!}{i!} q_{j+1} \\
&= \frac{j!}{i!}\left(-\sum_{k=0}^{i-1} \frac{i!}{k!} R^n(i,k) - \sum_{k=i}^{j-1} q_k - (j+1)\,q_j + (j+1)\,q_{j+1} \right).
\end{aligned}$$

Hence $R^{n+1}(i,j)$ is divisible by $\frac{j!}{i!}$ for all $j \geq i$. □

Corollary 6.4. *For all $n \geq 1$ and $j \geq 0$, the $(0,j)$th entry of R^n is divisible by $j!$.*

The most important feature of R^n is that the nth complementary Bell number is found in the top left entry. This is how the matrix is related to the complementary Bell numbers.

Theorem 6.5. *For all $n \in \mathbb{N}$, $\widetilde{B}(n) = R^n(0,0)$.*

Proof. Let $n \in \mathbb{N}$. By Corollary 3.2, $\widetilde{B}(n) = \lambda_n(0)$. By Equation 3.2,

$$\begin{aligned}
\lambda_n(x) &= \sum_{k=0}^{n} a_n(k) x^{(k)} \\
\text{so } \lambda_n(0) &= \sum_{k=0}^{n} a_n(k) 0^{(k)} \\
&= a_n(0).
\end{aligned}$$

By Theorem 5.3, $\mathbf{a}_n = R\mathbf{a}_{n-1} = R^2\mathbf{a}_{n-2} = \cdots = R^n\mathbf{a}_0$. Since $\lambda_0(x) = 1$, its coefficient vector $\mathbf{a}_0 = (1,0,0,\ldots)$, thus $R^n\mathbf{a}_0$ is the first column of R^n. Hence $\widetilde{B}(n) = \lambda_n(0) = a_n(0) = R^n(0,0)$. □

Of course, dealing with infinite matrices is difficult, so we want to be able to replace R with a finite matrix. For $m \geq 1$, define R_m to be the finite matrix that consists of the upper left $2^m \times 2^m$ block of R. Thus for each $0 \leq i \leq 2^m - 1$ and for each $0 \leq j \leq 2^m - 1$, $R_m(i,j) = R(i,j)$.

The following proposition will allow us to replace the infinite matrix R^n with a finite matrix R^n_m, provided we work modulo 2^{2^m-1}.

Proposition 6.6. *For all $m, n \geq 1$ and for each $0 \leq j \leq 2^m - 1$,*

$$R^n_m(0,j) \equiv R^n(0,j) \mod 2^{2^m-1}.$$

Proof. Let $m \geq 1$. The proof will proceed by induction on n. If $n = 1$, then $R^1_m(0,j) = R^1(0,j)$ for each $0 \leq j \leq 2^m - 1$ by definition. Now assume that the lemma holds for some $n \in \mathbb{N}$. The $(0,j)$th entry of R^{n+1}_m is given by

$$R^{n+1}_m(0,j) = \sum_{k=0}^{2^m-1} R^n_m(0,k)R_m(k,j) \equiv \sum_{k=0}^{2^m-1} R^n(0,k)R(k,j) \mod 2^{2^m-1} \tag{6.1}$$

by the definition of R_m and by the induction hypothesis.

If $j < 2^m - 1$, then $R(k,j) = 0$ for all $k \geq 2^m$ by the definition of R. Hence the $(0,j)$th entry of R^{n+1} is given by

$$R^{n+1}(0,j) = \sum_{k=0}^{\infty} R^n(0,k)R(k,j) = \sum_{k=0}^{2^m-1} R^n(0,k)R(k,j) \equiv R^{n+1}_m(0,j) \mod 2^{2^m-1}$$

by Equation 6.1.

If $j = 2^m - 1$, then $R(2^m, 2^m - 1) = 1$ and $R(k, 2^m - 1) = 0$ for all $k \geq 2^m + 1$. Hence the $(0, 2^m - 1)$th entry of R^{n+1} is given by

$$\begin{aligned} R^{n+1}(0,j) &= \sum_{k=0}^{\infty} R^n(0,k)R(k,j) \\ &= \sum_{k=0}^{2^m-1} R^n(0,k)R(k,j) + R^n(0,2^m)R(2^m, 2^m-1) \\ &= \sum_{k=0}^{2^m-1} R^n(0,k)R(k,j) + R^n(0,2^m) \\ &\equiv R^{n+1}_m(0,j) + R^n(0,2^m) \mod 2^{2^m-1} \end{aligned} \tag{6.2}$$

by Equation 6.1. By Corollary 6.4, $R^n(0,2^m)$ is divisible by $(2^m)!$, so $R^n(0,2^m) = (2^m)!q$ for some integer $q \geq 0$. By Legendre's formula, the 2-adic valuation of $(2^m)!$ equals $\dfrac{2^m - s_2(2^m)}{2-1} = 2^m - 1$, where $s_2(2^m)$ is the sum of the digits of the base 2 expansion of 2^m. Hence $(2^m)! = 2^{2^m-1}r$ where r is an odd integer. Then continuing from Equation 6.2, we have

$$\begin{aligned} R^{n+1}(0,j) &\equiv R^{n+1}_m(0,j) + R^n(0,2^m) \\ &\equiv R^{n+1}_m(0,j) + 2^{2^m-1}qr \\ &\equiv R^{n+1}_m(0,j) \mod 2^{2^m-1}. \end{aligned}$$

Therefore by induction the lemma holds for all $n \in \mathbb{N}$. □

The following results parallel Proposition 7.7 and Corollary 7.8 of [1].

Corollary 6.7. *For all $n \in \mathbb{N}$, $\widetilde{B}(n) \equiv R^n_m(0,0) \mod 2^{2^m-1}$.*

Proof. By Theorem 6.5 and Proposition 6.6,

$$\widetilde{B}(n) = R^n(0,0) \equiv R_m^n(0,0) \mod 2^{2^m-1}.$$

□

Corollary 6.8. *For all $n, k \in \mathbb{N}$,*

$$\widetilde{B}(n+k) \equiv \sum_{j=0}^{2^m-1} R_m^n(0,j) R_m^k(j,0) \mod 2^{2^m-1}.$$

Lemma 6.9. *If $m \geq 2$, $R_m^2(0,0) = 0$, $R_m^2(1,0) = -3$, $R_m^2(2,0) = 1$, and $R_m^2(r,0) = 0 \ \forall r \geq 3$*

Proof. Using induction and Lemma 3:

$$\begin{aligned} R_{m+1}^2 &= \left[\left(\begin{array}{c|c} R_m & 0 \\ \hline V_m & R_m \end{array}\right) + 2^m \left(\begin{array}{c|c} 0 & * \\ \hline 0 & * \end{array}\right)\right]\left[\left(\begin{array}{c|c} R_m & 0 \\ \hline V_m & R_m \end{array}\right) + 2^m \left(\begin{array}{c|c} 0 & * \\ \hline 0 & * \end{array}\right)\right] \\ &= \left(\begin{array}{c|c} R_m^2 & 0 \\ \hline V_m R_m + R_m V_m & * \end{array}\right) + 2^m \left(\begin{array}{c|c} 0 & * \\ \hline 0 & * \end{array}\right) + \left(\begin{array}{c|c} B_m V_m & * \\ \hline D_m V_m & * \end{array}\right) + 2^{2m} \left(\begin{array}{c|c} 0 & * \\ \hline 0 & * \end{array}\right) \end{aligned}$$

$$(B_m V_m)(r,0) = \sum_s B_m(r,s) V_m(s,0) = 0$$

and similarly, the first column of $V_m P_m + P_m V_m$ and $D_m V_m$ is 0
So $R_{m+1}^2(r,0) = R_m^2(r,0)$ for $0 \leq r \leq 2^{m+1} - 1$ □

7. Conclusion

We have constructed an infinite matrix R that is analogous to the matrix P constructed by T. Amdeberhan, V. De Angelis, and V.H. Moll, and using this matrix have paralleled many of the results found using the P matrix. We believe that given more time, we could find parallel results to all of the results found for the P matrix. Hence using the R matrix we would be able to paralell the proof in [1] that $\widetilde{B}(n) = 0$ for at most one integer $n > 2$.

References

[1] T. Amdeberhan, V. De Angelis, and V.H. Moll. Complementary Bell Numbers: Arithmetical Properties and Wilf's Conjecture. 2011.

Mathematics Department, University of Mississippi, University, Mississippi

Mathematics Department, University of Louisiana at Lafayette, Lafayette, Louisiana

Mathematics Department, Louisiana State University, Baton Rouge, Louisiana
E-mail address: sseger4@lsu.edu