Umbral Calculus and the Boustrophedon Transform

Daniel Berry, Jonathan Broom, Dewayne Dixon, Adam Flaherty

Friday July 12, 2013

Contents

Introduction

The boustrophedon transform of a sequence, a_n, produces a sequence b_n by populating a triangle in the following manner:

$$
\begin{tikzcd}
 & & & & a_0 \arrow[dr] & & & & \\
 & & & a_1 \arrow[rr] \arrow[dl] & & b_1 \arrow[dl] & & & \\
 & & b_2 \arrow[dr] & & T_{(2,1)} \arrow[ll] \arrow[dr] & & a_2 \arrow[ll] \arrow[dr] & & \\
 & a_3 \arrow[rr] \arrow[dl] & & T_{(3,1)} \arrow[rr] \arrow[dl] & & T_{(3,2)} \arrow[rr] \arrow[dl] & & b_3 \arrow[dl] & \\
b_4 & & T_{(4,3)} \arrow[ll] & & T_{(4,2)} \arrow[ll] & & T_{(4,1)} \arrow[ll] & & a_4 \arrow[ll]
\end{tikzcd}
$$

Introduction 2

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The transform can be defined more formally using a recurrence relation. Let the numbers $T_{k,n}$ $(k \geq n \geq 0)$ be defined

$$T_{n,0} = a_n$$

$$T_{k,n} = T_{k,n-1} + T_{k-1,k-n}(k \geq n > 0).$$

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$$T_{n,0} = a_n$$

$$T_{k,n} = T_{k,n-1} + T_{k-1,k-n} (k \geq n > 0).$$

Now,

$$b_n = T_{n,n}$$

Contents

Graphical Interpretation

We can think of the boustrophedon triangle as a directed graph. Using this interpretation, we construct a bijection between the set of paths beginning at $T_{0,0}$ and ending at $T_{n,n}$ and the set of alternating permutation on $[n]$.

Path-Permutation Bijection Theorem

Theorem

Let $\pi(n, n, 0)$ be the set of paths starting at $(0, 0)$ and ending at (n, n). Then there exists a bijection $\phi : \pi(n, n, 0) \rightarrow DU(n)$.

Constructing the Bijection

Given a path, we can construct a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ using:
σ_{2i} is the $f(n-2j+1)^{th}$ element from the left (with the arrows) of $[n] \setminus \{\sigma_1, \sigma_2, \cdots, \sigma_{2j-1}\}$
and
σ_{2i+1} is the $f(n-2j)^{th}$ element from the right (against the arrows) of $[n] \setminus \{\sigma_1, \sigma_2, \cdots, \sigma_{2j-1}\}$ where $0 < i \leq n$. This is illustrated in later examples.

Inverting the Bijection

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- To map a permutation to a path: Given a $\sigma = \sigma_1\sigma_2\cdots\sigma_j\cdots\sigma_n \in DU(n)$, the set of pairs $\{(k, f(k))\}$ where

$$f(n-2j) = n+1-|\{\sigma_i : \sigma_i > \sigma_{2j+1}, i < 2j\}| - \sigma_{2j+1}$$
$$f(n-2j-1) = \sigma_{2j+2} - |\{\sigma_i : \sigma_i < \sigma_{2j+2}, i < 2j+1\}|$$

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Example 1

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$$0 \leq j < \frac{n+1}{2}, j \in \mathbb{Z}$$

to determine the vertex where the path enters the $k-th$ row, where $k = n-2j$ or $k = n-2j-1$

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\begin{aligned}
f(9-2(0)) &= f(9) = 9+1-0-3 &&= 7\\
f(9-2(0)-1) &= f(8) = 1-0 &&= 1\\
f(9-2(1)) &= f(7) = 9+1-0-6 &&= 4\\
f(9-2(1)-1) &= f(6) = 2-1 &&= 1\\
f(9-2(2)) &= f(5) = 9+1-0-7 &&= 3\\
f(9-2(2)-1) &= f(4) = 4-3 &&= 1\\
f(9-2(3)) &= f(3) = 9+1-0-9 &&= 1\\
f(9-2(3)-1) &= f(2) = 5-4 &&= 1\\
f(9-2(4)) &= f(1) = 9+1-1-8 &&= 1
\end{aligned}
$$

Example 1

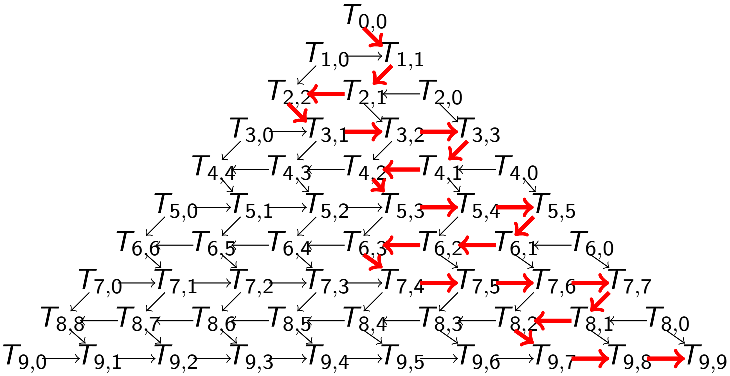

Figure: The path corresponding to Example 1

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\sigma_1 &= \text{the } f(9)^{th} \text{from the right of} & \{1,2,3,4,5,6,7,8,9\} &= 3\\
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\sigma_3 &= \text{the } f(7)^{th} \text{from the right of} & \{2,4,5,6,7,8,9\} &= 6\\
\sigma_4 &= \text{the } f(6)^{th} \text{from the left of} & \{2,4,5,7,8,9\} &= 2\\
\sigma_5 &= \text{the } f(5)^{th} \text{from the right of} & \{4,5,7,8,9\} &= 7\\
\sigma_6 &= \text{the } f(4)^{th} \text{from the left of} & \{4,5,8,9\} &= 4\\
\sigma_7 &= \text{the } f(3)^{th} \text{from the right of} & \{5,8,9\} &= 9\\
\sigma_8 &= \text{the } f(2)^{th} \text{from the left of} & \{5,8\} &= 5\\
\sigma_9 &= \text{the } f(1)^{th} \text{from the right of} & \{8\} &= 8
\end{aligned}$$

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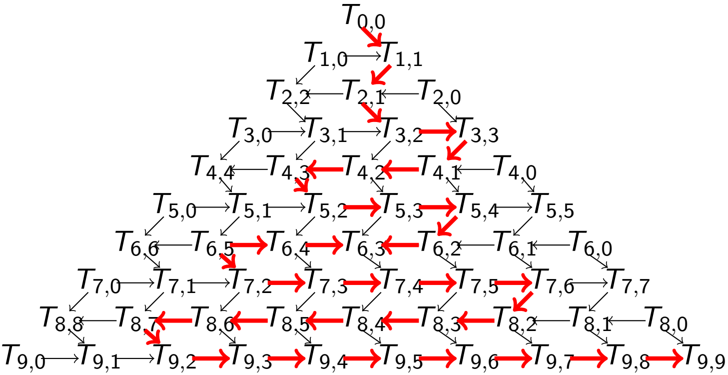

Figure: The path corresponding to Example 2

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- Output sequence:
  $1, 2, 4, 10, 32, 122, 544, 2770, 15872, 101042, \dots$

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- It may be possible to develop a general expression for the elements of $T(n, k)$ as a linear combination of the Euler numbers

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$$A(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$$

- By making the substitution of a_n to a^n, we get

$$\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n = \sum_{n=0}^{\infty} \frac{a^n}{n!} x^n = e^{ax}$$

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- A is called the **umbra** of (a_n).

Umbral Substitution

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- A **linear functional** is a map $L : V \to \mathbb{F}$ from a vector space V into its field of scalars $\mathbb{F}$ for which

$$L(cu + v) = cL(u) + L(v)$$

for all $u, v \in V$ and $c \in \mathbb{F}$.

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$$L(A^n) = a_n$$
- Since $\{A^n \mid n \geq 0\}$ is a basis of $\mathbb{R}[A]$, this defines L on the whole space.

Umbral Substitution for Multiple Sequences

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- $L(A^n B^m) = L(A^n)L(B^m)$

Umbral Calculus for a Sequence Transformation

Umbral Calculus for a Sequence Transformation

Proposition

Let (a_n) and (c_n) be fixed sequences of real numbers and define a new sequence (s_n) by the transformation

$$s_n = \sum_{k=0}^{n} \binom{n}{k} a_k c_{n-k}$$

Then $L(A^n) = L((S - C)^n)$ for all $n \geq 1$.

Inverse Transform

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Proposition

The inverse of the transformation is given by the equation

$$a_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} s_k c_{n-k}$$

Umbral Calculus for the Boustrophedon Transform

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Umbral Calculus for a Sequence Transformation

Corollary

The inverse of the boustrophedon transform is given by the equation

$$a_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} b_k E_{n-k}$$

for $n \geq 1$.

Umbral Calculus for a Sequence Transformation

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The inverse of the boustrophedon transform is given by the equation

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for $n \geq 1$.

Proof.

Take the sequence (c_n) to be the Euler numbers (E_n) in Proposition 2. □

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- What other sequences have boustrophedon transforms of combinatorial importance?
- Are there other interesting sequence transformations with properties similar to the boustrophedon transform?

Contents

Thanks

- LSU
- Our Parent Universities
- SMILE @ LSU and Dr. Davidson
- The NSF
- Dr. DeAngelis
- Our wonderful mentor Emily McHenry

Contents

Works Cited

- Miller, J. & Sloane, N.J.A. & Young, N.E., *A New Operation on Sequences: The Boustrophedon Transform**, J. Combinatorial Theory, Series A, volume 76, pages 44-54, 1996.
- Roman, S. & Rota, G., *The Umbral Calculus*, Volume 27, Number 2, pages 95-120. New York & London: Academic Press, 1978.