AN EPIDEMIOLOGICAL MODEL OF HIV

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Abstract. The mathematical modeling of an infectious disease is important to understanding the way a disease spreads. Human Immunodeficiency Virus (HIV) is a widespread problem that affects an estimated 33.3 million people, including 2.5 million children. The purpose of this paper is to determine the conditions under which the HIV disease persists and the conditions under which it dies out within a population. We also wish to examine the stability of these conditions, which will allow us to see how changes in the parameters affect the validity of our determined conditions.

1. Introduction

The simplest epidemiological models rely on the infected class and the susceptible class. The susceptible class consists of the individuals in the population who have the potential to be infected, whereas the infected class contains those already infected. Our model uses these classes, but also breaks down the infected class into two categories: those with HIV and those with Acquired Immunodeficiency Syndrome (AIDS).

For this model of HIV/AIDS, some key assumptions are made:

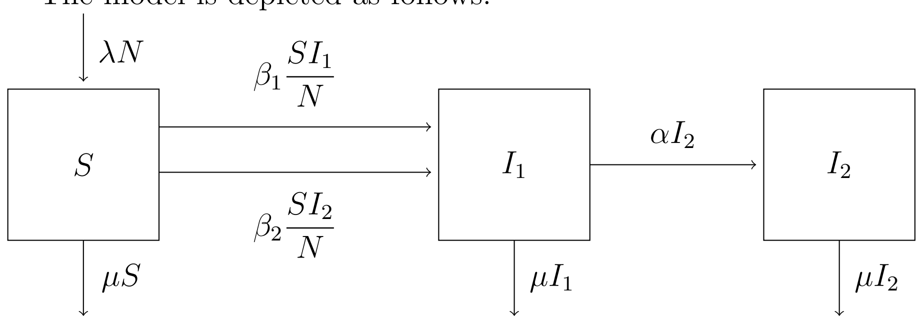
- No vertical (mother-to-child) transmission.
- The disease naturally progresses from HIV to AIDS.
- The difference between the natural death rate and the death rate from AIDS is negligible. We thus only consider the natural death rate.

This model also assumes that susceptible individuals infected by people with AIDS progress into the HIV class, not directly into the AIDS class.

To better understand the spread of the disease, equilibrium points were found where the differential equations were equal to zero (implying no net change in any of the classes). The disease free equilibrium point (DFE) is the value to which the three populations converge under the condition that there are no infected individuals within the population. This implies that every person in the population is in the susceptible class. The disease endemic equilibrium point (DEE) is the value to which the three populations converge under the condition that the disease persists in the population. Next, a threshold parameter, R_0, was found. R_0 represents the average number of secondary cases that a single infected individual produces when introduced into a completely susceptible population. When of $R_0 < 1$, the disease will gradually become extinct. When $R_0 > 1$, the disease will persist.

2. The Model

The model is depicted as follows:



- λ is the birth rate of the entire population.
- μ is the natural death rate.
- β_1 is the rate of infection caused by susceptibles coming into contact with those infected with HIV.
- β_2 is the rate of infection caused by susceptibles coming into contact with those infected with AIDS.
- α is the rate of progression of the disease from HIV to AIDS.

3. Finding the DFE and DEE points

To find the Disease Free Equilibrium (DFE) point and the Disease Endemic Equilibrium (DEE) point, we use the model to begin writing the differential equations for each population class. These equations represent the movement of people in and out of the population classes as time progresses.

$$\frac{dS}{dt} = \lambda N - \mu S - \beta_1 \frac{SI_1}{N} - \beta_2 \frac{SI_2}{N}$$

$$\frac{dI_1}{dt} = \beta_1 \frac{SI_1}{N} + \beta_2 \frac{SI_2}{N} - \alpha I_1 - \mu I_1$$

$$\frac{dI_2}{dt} = \alpha I_1 - \mu I_2$$

Note that $S + I_1 + I_2 = N$, so

$$\frac{dN}{dt} = (\lambda - \mu)N.$$

Because the population is not constant, we must make a change of variable.

$$s = \frac{S}{N} \quad i_1 = \frac{I_1}{N} \quad i_2 = \frac{I_2}{N}$$

$$s + i_1 + i_2 = 1$$

We now have three new differential equations with which we will work.

$$\frac{ds}{dt} = \frac{1}{N}\frac{dS}{dt} - \frac{1}{N}\frac{dN}{dt}s = \lambda(1-s) - \beta_1 s i_1 - \beta_2 s i_2$$

$$\frac{di_1}{dt} = \frac{1}{N}\frac{dI_1}{dt} - \frac{1}{N}\frac{dN}{dt}i_1 = \beta_1 s i_1 + \beta_2 s i_2 - (\lambda + \alpha)i_1$$

$$\frac{di_2}{dt} = \frac{1}{N}\frac{dI_2}{dt} - \frac{1}{N}\frac{dN}{dt}i_2 = \alpha i_1 - \lambda i_2$$

To eliminate one of the equations and make the system easier to solve, we will work in terms of i_1 and i_2. Recall that $s = 1 - i_1 - i_2$.

We now begin solving for the DFE point and the DEE point. For the DFE point, there are no individuals moving from the susceptible class into the infected classes, that is to say, $\frac{ds}{dt} = 0$ and $i_1 = i_2 = 0$.

$$\frac{ds}{dt} = \lambda(1-s) = 0$$

$$s = 1$$

Our DFE point, in terms of (s, i_1), is $(1, 0)$. However, we will work in terms of (i_1, i_2), so our DFE point is $(0, 0)$.

To find the DEE point at which the disease persists, we set $\dfrac{di_1}{dt} = 0$ and $\dfrac{di_2}{dt} = 0$ and solve for i_1^* and i_2^*.

From $\frac{di_2}{dt}$, we get that

$$i_1^* = \frac{\lambda}{\alpha} i_2^*,$$

and from $\frac{di_1}{dt}$, we have that

$$i_1^* = \frac{-\beta_2 s^* i_2^*}{\beta_1 s^* - (\lambda + \alpha)}.$$

Therefore,

$$\frac{\lambda}{\alpha} i_2^* = \frac{-\beta_2 s^* i_2^*}{\beta_1 s^* - (\lambda + \alpha)}$$

$$\Longrightarrow \frac{\lambda}{\alpha}\beta_1 s^* - (\lambda + \alpha) = -\beta_2$$

$$\Longrightarrow s^* = \frac{\lambda^2 + \alpha\lambda}{\lambda\beta_1 + \alpha\beta_2}.$$

From $\frac{ds}{dt}$, we get

$$i_2^* = \frac{\lambda(1 - s^*)}{\beta_1 s^* \dfrac{\lambda}{\alpha} + \beta_2 s^*}.$$

Substituting in s^* here gives

$$i_2^* = \frac{\alpha}{\lambda + \alpha} - \frac{\lambda\alpha}{\lambda\beta_1 + \alpha\beta_2}.$$

Combining this with the above yields

$$i_1^* = \frac{\lambda}{\lambda + \alpha} - \frac{\lambda^2}{\lambda\beta_1 + \alpha\beta_2}.$$

$({i_1}^*, {i_2}^*)$ is the DEE point.

Thus $({i_1}^*, {i_2}^*) := (\dfrac{\lambda}{\lambda + \alpha} - \dfrac{\lambda^2}{\lambda\beta_1 + \alpha\beta_2}, \dfrac{\alpha}{\lambda + \alpha} - \dfrac{\lambda\alpha}{\lambda\beta_1 + \alpha\beta_2})$.

4. The Threshold parameter, R_0

The DEE point means that the disease exists and there are members of the population in the i_1 class. Setting $i_1 > 0$ gives

$$\frac{\lambda}{\lambda+\alpha} > \frac{\lambda^2}{\lambda\beta_1+\alpha\beta_2} \Longrightarrow \frac{\lambda\beta_1+\alpha\beta_2}{\lambda(\lambda+\alpha)} > 1.$$

In the following sections, we show that this value, $\dfrac{\lambda\beta_1+\alpha\beta_2}{\lambda(\lambda+\alpha)}$, R_0.

5. Proving local stability with the Jacobian

We now prove the local stability of our DFE point and DEE point by computing the trace and determinant of the Jacobian matrix.

Let

$$f(i_1,i_2) := \frac{di_1}{dt} = \beta_1 s i_1 + \beta_2 s i_2 - (\lambda+\alpha)i_1$$

$$= \beta_1(1-i_1-i_2)i_1 + \beta_2(1-i_1-i_2)i_2 - (\lambda+\alpha)i_1$$

and

$$g(i_1,i_2) := \frac{di_2}{dt} = \alpha i_1 - \lambda i_2.$$

We then have

$$J = \begin{bmatrix} \dfrac{\delta f}{\delta i_1} & \dfrac{\delta f}{\delta i_2} \\[2ex] \dfrac{\delta g}{\delta i_1} & \dfrac{\delta g}{\delta i_2} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{matrix} -\beta_1 i_1 - \beta_2 i_2 - (\lambda+\alpha) \\ +(1-i_1-i_2)\beta_1 \end{matrix} & \begin{matrix} -\beta_1 i_1 - \beta_2 i_2 \\ +\beta_2(1-i_1-i_2) \end{matrix} \\[3ex] \alpha & -\lambda \end{bmatrix}$$

For the DFE point, we assume that $R_0 < 1$ and we evaluate J at $(0,0)$:

$$J = \begin{bmatrix} \dfrac{\delta f}{\delta i_1}(0,0) & \dfrac{\delta f}{\delta i_2}(0,0) \\[2ex] \dfrac{\delta g}{\delta i_1}(0,0) & \dfrac{\delta g}{\delta i_2}(0,0) \end{bmatrix} = \begin{bmatrix} \beta_1-\lambda-\alpha & \beta_2 \\ \alpha & -\lambda \end{bmatrix}$$

It follows that

$$Tr(J) = \beta_1 - 2\lambda - \alpha = -\lambda - \alpha + \beta_1 - \lambda$$

and

$$Det(J) = -\lambda(\beta_1 - \lambda - \alpha) - \alpha\beta_2 = -\lambda\beta_1 - \alpha\beta_2 + \lambda(\lambda + \alpha).$$

Because we know that

$$R_0 = \frac{\lambda\beta_1 + \alpha\beta_2}{\lambda(\lambda + \alpha)} < 1,$$

we have,

$$\frac{\lambda\beta_1 + \alpha\beta_2}{\lambda} < (\lambda + \alpha) \Longrightarrow -\lambda - \alpha < -\beta_1 - \frac{\alpha\beta_2}{\lambda}.$$

Using this with $Tr(J)$, we see that

$$-\lambda - \alpha + \beta_1 - \lambda < -\beta_1 - \frac{\alpha\beta_2}{\lambda} + \beta_1 - \lambda$$

$$= -\frac{\alpha\beta_2}{\lambda} < 0$$

$$\Longrightarrow Tr(J) < 0.$$

From $R_0 < 1$, we also know

$$\lambda\beta_1 + \alpha\beta_2 < \lambda(\lambda + \alpha)$$

$$-\lambda\beta_1 - \alpha\beta_2 > -\lambda(\lambda + \alpha).$$

Using this with $Det(J)$, we see that

$$-\lambda\beta_1 - \alpha\beta_2 + \lambda(\lambda + \alpha) > 0 \Longrightarrow Det(J) > 0$$

When the $Tr(J) < 0$ and $Det(J) > 0$, a sink is produced. A sink describes a locally asymptotically stable point because it means that the eigenvalues of our 2x2 matrix are both negative. It is well known that when both eigenvalues are negative, the general solution of the differential equation vanishes. Therefore, the DFE point is locally asymptotically stable.

We thus have the Jacobian,

$$J(s, i_1, i_2) = \begin{bmatrix} \begin{array}{c} \beta_1(1-i_1-i_2) - \beta_1 i_1 \\ -(\lambda+\alpha) \end{array} & \begin{array}{c} \beta_2(1-i_1-i_2) \\ -i_1\beta_1 - i_2\beta_2 \end{array} \\ \\ \alpha & -\lambda \end{bmatrix}$$

$$\begin{aligned} (1-i_1-i_2) &= \frac{\lambda^2+\lambda\alpha}{\lambda\beta_1+\alpha\beta_2} \\ i_1 &= \frac{\lambda}{\lambda+\alpha} - \frac{\lambda^2}{\lambda\beta_1+\alpha\beta_2} \\ i_2 &= \frac{\alpha}{\lambda+\alpha} - \frac{\alpha\lambda}{\lambda\beta_1+\alpha\beta_2} \end{aligned}$$

From this, we get $Tr(J)$.

$$\begin{aligned} \mathrm{Tr(J)} &= \beta_1\lambda\left(\frac{\lambda+\alpha}{\lambda\beta_1+\alpha\beta_2}\right) - (\lambda+\alpha) - \beta_1\lambda\left(\frac{1}{\lambda+\alpha} - \frac{\lambda}{\lambda\beta_1+\alpha\beta_2} - \lambda\right) \\ &= \beta_1\lambda - \left[\left(\frac{\lambda+\alpha}{\lambda\beta_1+\alpha\beta_2}\right)\left(\frac{\lambda+\alpha}{\lambda+\alpha}\right)\right] - \left[(\lambda+\alpha)\frac{(\lambda+\alpha)(\lambda\beta_1+\alpha\beta_2)}{(\lambda+\alpha)(\lambda\beta_1+\alpha\beta_2)}\right] \\ &- \beta_1\lambda\left[\left(\frac{1}{\lambda+\alpha}\right)\left(\frac{\lambda\beta_1+\alpha\beta_2}{\lambda\beta_1+\alpha\beta_2}\right) - \left(\frac{\lambda}{\lambda\beta_1+\alpha\beta_2}\right)\left(\frac{\lambda+\alpha}{\lambda+\alpha}\right)\right] - \lambda \\ &= \frac{\beta_1\lambda(\lambda+\alpha)^2}{(\lambda\beta_1+\alpha\beta_2)(\lambda+\alpha)} - \frac{(\lambda+\alpha)^2(\lambda\beta_1+\alpha\beta_2)}{(\lambda\beta_1+\alpha\beta_2)(\lambda+\alpha)} \\ &- \beta_1\lambda\left[\frac{(\lambda\beta_1+\alpha\beta_2)-\lambda(\lambda+\alpha)}{(\lambda\beta_1+\alpha\beta_2)(\lambda+\alpha)}\right] - \lambda \\ &= (\lambda+\alpha)^2\left[\frac{\cancel{\beta_1\lambda}}{(\lambda\beta_1+\alpha\beta_2)(\lambda+\alpha)} - \frac{(\cancel{\lambda\beta_1}+\alpha\beta_2)}{(\lambda\beta_1+\alpha\beta_2)(\lambda+\alpha)}\right] \\ &- \beta_1\lambda\left[\frac{(\lambda\beta_1+\alpha\beta_2)-\lambda(\lambda+\alpha)}{(\lambda\beta_1+\alpha\beta_2)(\lambda+\alpha)}\right] - \lambda \\ &< 0 \end{aligned}$$

Also from the Jacobian, we get $Det(J)$.

$$\begin{aligned}
\text{Det(J)} &= \lambda\beta_1\left(\frac{\lambda}{\lambda+\alpha}-\frac{\lambda^2}{\lambda\beta_1+\alpha\beta_2}\right) \\
&- \lambda\left[1-\left(\frac{\lambda}{\lambda+\alpha}-\frac{\lambda^2}{\lambda\beta_1+\alpha\beta_2}\right)-\left(\frac{\alpha}{\lambda+\alpha}\frac{\alpha\lambda}{\lambda\beta_1+\alpha\beta_2}\right)\right] \\
&+ \lambda\alpha\beta_2\left(\frac{\lambda}{\lambda+\alpha}-\frac{\lambda\alpha}{\lambda\beta_1+\alpha\beta_2}\right)+\lambda(\lambda+\alpha) \\
&+ \lambda\alpha\beta_1\left(\frac{\lambda}{\lambda+\alpha}-\frac{\lambda^2}{\lambda\beta_1+\alpha\beta_2}\right) \\
&- \alpha\beta_2\left(1-\frac{\lambda}{\lambda+\alpha}+\frac{\lambda^2}{\lambda\beta_1+\alpha\beta_2}-\frac{\alpha}{\lambda+\alpha}+\frac{\alpha\lambda}{\lambda\beta_1+\alpha\beta_2}\right) \\
&+ \alpha^2\beta_2\left(\frac{1}{\lambda+\alpha}-\frac{\lambda}{\lambda\beta_1+\alpha\beta_2}\right) \\
&= \lambda\beta_1\left(\frac{\lambda}{\lambda+\alpha}-\frac{\lambda^2}{\lambda\beta_1+\alpha\beta_2}\right) \\
&- \lambda\left[1-\lambda\left(\frac{1}{\lambda+\alpha}-\frac{\lambda}{\lambda\beta_1+\alpha\beta_2}\right)-\alpha\left(\frac{1}{\lambda+\alpha}-\frac{\lambda}{\lambda\beta_1+\alpha\beta_2}\right)\right] \\
&+ \lambda\alpha\beta_2\left(\frac{1}{\lambda+\alpha}-\frac{\lambda}{\lambda\beta_1+\alpha\beta_2}\right)+\lambda(\lambda+\alpha) \\
&+ \lambda\alpha\beta_1\left(\frac{\lambda}{\lambda+\alpha}-\frac{\lambda^2}{\lambda\beta_1+\alpha\beta_2}\right) \\
&- \alpha\beta_2\left[1-\lambda\left(\frac{1}{\lambda+\alpha}-\frac{\lambda}{\lambda\beta_1+\alpha\beta_2}\right)-\alpha\left(\frac{1}{\lambda+\alpha}-\frac{\lambda}{\lambda\beta_1+\lambda\beta_2}\right)\right] \\
&+ \alpha\beta_2\left(\frac{1}{\lambda+\alpha}-\frac{\lambda}{\lambda\beta_1+\alpha\beta_2}\right)-\alpha^2\beta_2-\lambda\left(\frac{1}{\lambda+\alpha}-\frac{\lambda}{\lambda\beta_1+\alpha\beta_2}\right)
\end{aligned}$$

$$\begin{aligned}
D &= \lambda\left[1-\left(\frac{1}{\lambda+\alpha}-\frac{\lambda}{\lambda\beta_1+\alpha\beta_2}\right)(\lambda+\alpha)\right]-\alpha\beta_2\left[1-\left(\frac{1}{\lambda+\alpha}\right)\right. \\
&- \left.\left(\frac{\lambda}{\lambda\beta_1+\alpha\beta_2}\right)(\lambda+\alpha)\right] \\
0 &< \lambda\left[\frac{\lambda(\lambda+\alpha)}{\lambda\beta_1+\alpha\beta_2}\right]-\alpha\beta_2\left[\frac{-\lambda(\lambda+\alpha)}{\lambda\beta_1+\alpha\beta_2}\right]
\end{aligned}$$

So $Det(J) > 0$ and $Tr(J) < 0$. These conditions also produce a sink, so our DEE point is locally asymptotically stable.

6. Proving Global Stability with the Lyapunov

To prove points are not just locally stable, but also globally stable, we use a Lyapunov function, defined below.

Theorem 6.1. *[1] Let x^* be an equilibrium point for $X' = f(x)$. If there exists a differentiable function $L : U \to \mathbb{R}$ defined on an open set U containing x^* such that:*

(i) *$L(x^*) = 0$ and $L(x) > 0$ if $x \neq x^*$;*

(ii) *$\frac{dL}{dt} < 0$ in $U \backslash x^*$*

Then x^* is globally asymptotically stable.

We will prove the stability of the DFE point. To do this, let

$$L(i_1, i_2) := i_1 + \frac{\beta_2}{\lambda} i_2.$$

It is clear that at $(0,0)$, $L(i_1, i_2)$ satisfies condition one, and so we have only to show condition two. Consider the following,

$$\frac{dL}{dt} = \frac{di_1}{dt} + \frac{\beta_2}{\lambda}\frac{di_2}{dt} = \beta_1(1-i_1-i_2)i_1 + \beta_2(1-i_1-i_2)i_2 - (\lambda+\alpha)i_1 + \frac{\beta_2}{\lambda}(\alpha i_1 - \lambda i_2)$$

$$= i_1(\beta_1(1-i_1-i_2) - (\lambda+\alpha)) + \beta_2(1-i_1-i_2)i_2 + \frac{\beta_2 \alpha i_1}{\lambda} - \beta_2 i_2.$$

Because we know $R_0 < 1$, it follows that

$$< i_1(\beta_1 - (\lambda+\alpha)) + \beta_2(1-i_1-i_2)i_2 + \frac{\beta_2 \alpha i_1}{\lambda} - \beta_2 i_2$$

$$= \frac{i_1}{\lambda}(\beta_1\lambda + \alpha\beta_2 - (\lambda+\alpha)\lambda) + \beta_2(1-i_1-i_2)i_2 + \frac{\beta_2 \alpha i_1}{\lambda} - \beta_2 i_2 - \frac{\beta_2 \alpha i_1}{\lambda}$$

$$= \frac{i_1}{\lambda}(\beta_1\lambda + \alpha\beta_2 - (\lambda+\alpha)\lambda) + \beta_2(1-i_1-i_2)i_2 - \beta_2 i_2 < 0.$$

We thus conclude that our DFE point is globally stable.

7. Parameter Simulations

To visualize the threshold parameter, we use MatLab to simulate our data by inputing certain values for λ, β_1, β_2, μ and α. These values either produced $R_0 > 1$ or $R_0 < 1$. The resulting figures are below. s is represented by the dotted line, i_1 is represented by the solid curve and i_2 is represented by the dashed curve.

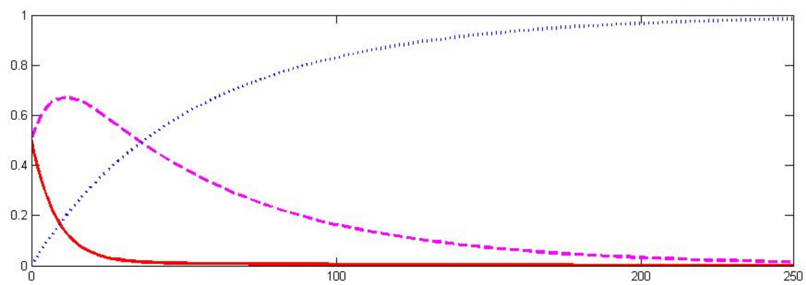


Figure 1. *This figure represents the DFE point. It shows the proportions of the population in the I_1 and I_2 classes with respect to time when $R_0 < 1$. The sum of the number of people in I_1 and I_2 approaches zero, and the number of people in s approaches 100%, meaning the entire population is in the susceptible class and the disease becomes extinct.*

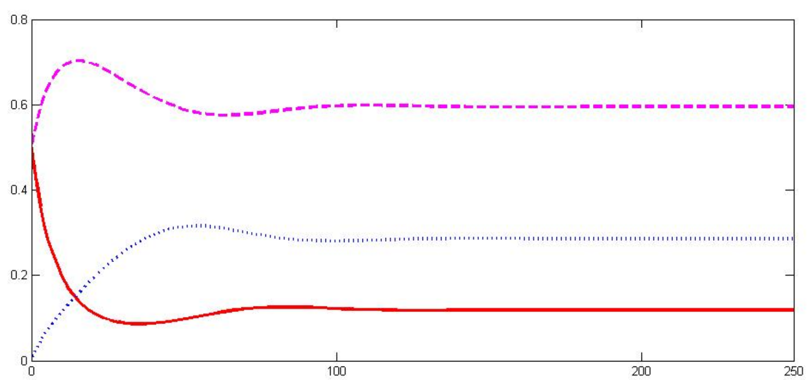
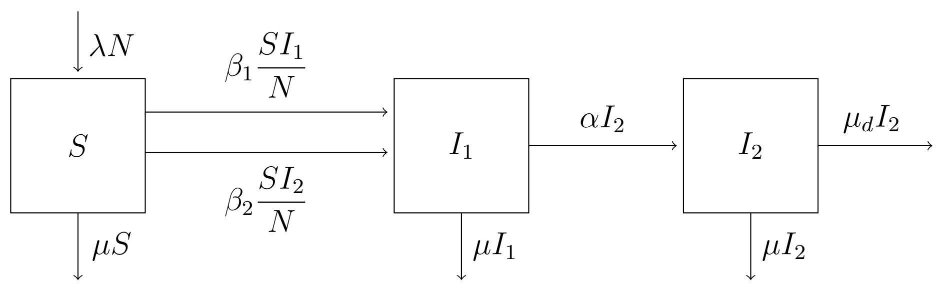


Figure 2. *This figure represents the DEE point. It shows the proportions of the population in the I_1 and I_2 classes with respect to time when $R_0 > 1$. The sum of the number of people in I_1 and I_2 approaches 100%, meaning the entire population is in the infected class and the disease persists.*

8. Additional Considerations

If we were to include a death rate from AIDS separate from the natural death rate, our model would change.

8.1. The Revised Model. The revised model is depicted as follows:



- $\mu_d I_2$ is the death rate of people who have AIDS.

To find the Disease Free Equilibrium (DFE) point and the Disease Endemic (DE) point, we use the model to begin writing the differential equations for each population class. These equations represent the movement of people in and out of the population classes as time progresses.

$$\frac{dS}{dt} = \lambda N - \mu S - \beta_1 \frac{SI_1}{N} - \beta_2 \frac{SI_2}{N}$$

$$\frac{dI_1}{dt} = \beta_1 \frac{SI_1}{N} + \beta_2 \frac{SI_2}{N} - \alpha I_1 - \mu I_1$$

$$\frac{dI_2}{dt} = \alpha I_1 - \mu I_2 - \mu_d I_2$$

Note that $S + I_1 + I_2 = N$, so

$$\frac{dN}{dt} = (\lambda - \mu)N - \mu_d I_2.$$

Because the population is not constant, we must make a change of variable.

$$s = \frac{S}{N} \quad i_1 = \frac{I_1}{N} \quad i_2 = \frac{I_2}{N}$$

$$s + i_1 + i_2 = 1$$

We now have three new differential equations with which we will work.

$$\frac{ds}{dt} = \frac{1}{N}\frac{dS}{dt} - \frac{1}{N}\frac{dN}{dt}s$$

$$= \lambda - \beta_1 s i_1 - \beta_2 s i_2 - \mu s - [((\lambda - \mu) - \mu_d i_2)]s$$

$$= \lambda - \beta_1 s i_1 - \beta_2 s i_2 - \mu s - \lambda s + \mu s + \mu_d s i_2$$

$$\frac{ds}{dt} = \lambda(1 - s) - \beta_1 s i_1 - \beta_2 s i_2 + \mu_d s i_2$$

Following in the same manner for i_1 and i_2 yields

$$\frac{di_1}{dt} = \beta_1 s i_1 + \beta_2 s i_2 - (\lambda + \alpha) i_1 + \mu_d i_1 i_2$$

$$\frac{di_2}{dt} = \alpha i_1 - (\lambda + \mu_d) i_2 + \mu_d {i_2}^2$$

To eliminate one of the equations and make the system easier to solve, we will work in terms of i_1 and i_2. Recall that $i_2 = 1 - s - i_1$.

We now begin solving for the DFE point and the DEE point. For the DFE point, there are no individuals moving from the susceptible class into the infected classes, that is to say, $\frac{ds}{dt} = 0$ and $i_1 = i_2 = 0$.

$$\frac{ds}{dt} = \lambda(1 - s) = 0$$

$$s = 1$$

Our DFE point is, in terms of (s, i_1) is $(1, 0)$. However, we will work in terms of (i_1, i_2), so our DFE point is $(0, 0)$.

To find the DEE point at which the disease goes extinct, we solve for i_1 and i_2 by setting $\frac{di_1}{dt} = 0$ and $\frac{di_2}{dt} = 0$.

From $\frac{di_2}{dt}$:

$$\begin{aligned}
i_1 &= \frac{(\lambda+\mu_d)i_2 - \mu_d i_2^2}{\alpha} \\
1-i_1 &= \frac{\alpha - (\lambda+\mu_d)i_2 + \mu_d i_2^2}{\alpha} \\
1-i_1-i_2 &= \frac{\alpha - (\lambda+\mu)i_2 + \mu_d i_2^2 - i_2^2}{\alpha} \\
s &= \frac{\alpha - (\lambda+\mu_d+\alpha)i_2 + \mu_d i_2^2}{\alpha}
\end{aligned}$$

Therefore, from the original equation of $\frac{di_1}{dt}$:

$$\begin{aligned}
\frac{di_1}{dt} &= \beta\left[\frac{\alpha - (\lambda+\mu_d+\alpha)i_2 + \mu_d i_2^2}{\alpha}\right]\left[\frac{(\lambda+\mu_d)i_2 - \mu_d i_2^2}{\alpha}\right] \\
&+ \beta_2\left[\frac{\alpha - (\lambda+\mu_d+\alpha)i_2 + \mu_d i_2^2}{\alpha}\right] i_2 - (\lambda+\alpha)\left[\frac{(\lambda+\mu_d)i_2 - \mu_d i_2^2}{\alpha}\right] \\
&+ \mu_d i_2\left[\frac{(\lambda+\mu_d)i_2 - \mu_d i_2^2}{\alpha}\right] = 0.
\end{aligned}$$

Thus,

$$\begin{aligned}
&= i_2^3(-\beta_1\mu_d^2) + i_2^2[\beta(\lambda+\mu_d)\mu_d + \beta_2\alpha\mu_d + \beta_1(\mu_d)(\lambda+\mu_d+\alpha) - \alpha\mu_d^2] \\
&= i_2[-\beta_1(\lambda+\mu_d)(\lambda+\mu_d+\alpha) - \beta_1(\mu_d\alpha) - \beta_2\alpha(\lambda+\mu_d+\alpha) + \alpha(\lambda+\alpha)\mu_d \\
&+ \alpha\mu_d(\lambda+\mu_d)] + \alpha\beta_1(\lambda+\mu_d) + \beta_2\alpha^2 - \alpha(\lambda+\alpha)(\lambda+\mu_d).
\end{aligned}$$

While this does not solve explicitly for i_1 and i_2, it is useful because it makes it possible to determine the conditions that guarantee the existence of i_2. The polynomial has two apparent sign changes, meaning there exist two or zero real roots by Descartes' rule of signs. We want the guaranteed existence of at least one real root, which would be the case if there were three sign changes, because complex roots only come in pairs. To make certain there are three sign changes, we want our final term $\beta_1(\lambda+\mu_d) + \beta_2\alpha - (\lambda+\alpha)(\lambda+\mu_d) < 0$. Meaning

$$\beta_1(\lambda+\mu_d) + \beta_2\alpha < (\lambda+\alpha)(\lambda+\mu_d)$$

$$\frac{\beta_1(\lambda+\mu_d) + \beta_2\alpha}{(\lambda+\alpha)(\lambda+\mu_d)} < 1$$

We have now found R_0 to be

$$\frac{\beta_1(\lambda+\mu_d)+\beta_2\alpha}{(\lambda+\alpha)(\lambda+\mu_d)}$$

and our DEE point to be $({i_1}^*, {i_2}^*)$.

Now we must prove the local stability of our DFE point and DEE point by computing the trace and determinant of the Jacobian matrix.

Let

$$f(i_1,i_2) = \frac{di_1}{dt} = \beta_1(1-i_1-i_2)i_1 + \beta_2(1-i_1-i_2)i_2 - (\lambda+\alpha)i_1 + \mu_d i_1 i_2$$

and

$$g(i_1,i_2) = \frac{di_2}{dt} = \alpha i_1 - (\lambda+\mu_d)i_2 + \mu_d {i_2}^2.$$

Then we have

$$J = \begin{bmatrix} \frac{\delta f}{di_1} & \frac{\delta f}{di_2} \\ \frac{\delta g}{di_1} & \frac{\delta g}{di_2} \end{bmatrix}$$

$$= \begin{bmatrix} \beta_1 - 2\beta_1 i_1 - \beta_1 i_2 - (\lambda+\alpha) + \mu_d i_2 & -\beta_1 i_1 + \beta_2 - \beta_2 i_1 - 2\beta_2 i_2 + \mu_d i_1 \\ \alpha & -\lambda - \mu_d + 2\mu_d i_2 \end{bmatrix}$$

For the DFE point, we get

$$J = \begin{bmatrix} \frac{\delta f}{di_1}(0,0) & \frac{\delta f}{di_2}(0,0) \\ \frac{\delta g}{di_1}(0,0) & \frac{\delta g}{di_2}(0,0) \end{bmatrix} = \begin{bmatrix} \beta_1 - \lambda - \alpha & \beta_2 \\ \alpha & -\lambda - \mu_d \end{bmatrix}$$

Which means we have

$$Tr(J) = \beta_1 - 2\lambda - \alpha - \mu_d$$

and

$$Det(J) = (\lambda + \alpha - \beta_1)(\lambda + \mu_d) - \alpha\beta_2.$$

Because we know that

$$R_0 = \frac{\beta_1(\lambda + \mu_d) + \beta_2\alpha}{(\lambda + \alpha)(\lambda + \mu_d)} < 1$$

for the disease to go extinct,

$$\beta_1(\lambda + \mu_d) + \beta_2\alpha < (\lambda + \alpha)(\lambda + \mu_d).$$

Note that because $\beta_2\alpha$ is positive, it can be removed from the above inequality and preserve the validity of the inequality.

$$\beta_1(\lambda + \mu_d) < (\lambda + \alpha)(\lambda + \mu_d)$$

$$\beta_1 < \lambda + \alpha$$

Recall

$$Tr(J) = \beta_1 - 2\lambda - \alpha - \mu_d.$$

$$= \beta_1 - (\lambda + \alpha) - (\mu_d + \alpha)$$

This implies that the $Tr(J) < 0$.

Recall

$$Det(J) = (\lambda + \alpha - \beta_1)(\lambda + \mu_d) - \alpha\beta_2$$
$$= -(\beta_1(\lambda + \mu_d) + \alpha\beta_2) + (\lambda + \alpha)(\lambda + \mu_d).$$

This implies that the $Det(J) > 0$.

With the $Tr(J) < 0$ and $Det(J) > 0$, a sink is produced. A sink describes a locally asymptotically stable point because it means that the eigenvalues (λ's) of our 2x2 matrix are both negative. As noted earlier, when both eigenvalues are negative, the general solution of the differential equation vanishes. Our DFE point is locally asymptotically stable.

9. Future work

Understanding the mathematical model of HIV is important to controlling the spread of the disease. We were able to find the necessary conditions (according to our model) for the disease to go extinct and for it to persist; we also were able to prove the stability of these conditions. This particular model of HIV took many things into account, but further study is necessary to make the model more accurate. For example, HIV progresses through three true stages of infection, although this model considered only two. Also, aside from the additional considerations we carried out, the death rates of infected individuals from the disease was regarded to be negligible and was included with the natural death rate.

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