EXPLORATION OF SPECIAL CASES OF LAPLACE TRANSFORMS

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Abstract. First of all this paper discusses details of the gamma function and explores some of its properties. Next, some of the properties of Laguerre functions are shown as well as how these functions relate to Laplace Transforms. In further study, differential equations and properties of Laplace transform will be used to calculate the Laplace transform of functions. Finally, many points of linear recursion relations will be explored and the Laplace transform will be used to solve them.

1. The Gamma Function

2. Laguerre Polynomials

Abstract. A polynomial p(x) in powers of x is a finite sum of terms like

$$p(x) = \sum_{k=0}^{n} a_k x^k \tag{2.1}$$

where k is a non-negative integer. The set of orthogonal polynomials contains polynomials that vanish when the product of any two different ones, multiplied by a function *w(x)*, called a weight function, are integrated over a certain interval. This makes it possible to expand an arbitrary function f(x) as a sum of the polynomials, each multiplied by a coefficient c(k), which is uniquely determined by integration. The Laguerre polynomials are orthogonal on the interval from 0 to ∞ with respect to the weight function w(x) = e^{-x} . They also have many interesting properties and identities some of which involve differential operators, recursion and integration. The Laplace transform is used to prove them here.

2.1. **Laguerre's equation.** The equation

$$ty'' + (1-t)y' + ny = 0, \tag{2.2}$$

where n is a nonnegative intger, is known as Laguerre's equation of order n. This differential equation possesses the polynomial solution

$$l_n(t) = \sum_{k=0}^{n} \frac{n!(-1)^k t^k}{(k!)^2(n-k)!} \tag{2.3}$$

The function $l_n(t)$ is known as the Laguerre polynomial of degree n. For n $\neq$ m,

$$\int_0^\infty e^{-t} l_n(t) l_m(t) dt = 0. \tag{2.4}$$

2.2. **Graph of Laguerre Polynomial.** Maple Heading 1 Maple Text Output Maple Dash Item Maple Bullet Item Maple Normal Maple Heading 4 Maple Heading 3 Maple Heading 2 Maple Warning Maple Title Maple Error Maple Hyperlink Maple 2D Math Maple Maple Input Maple 2D Output Maple 2D Input active2drestart; 1

active2dLaguerreL(3, t); 1

inert2dLaguerreL(3, t)

$$LaguerreL\,(3, t)$$

active2dl0 := simplify(inert2dl0 := 1-3*x+(3/2)*x$^2 - (1/6) * x^3$ l_3 $:= 1 - 3\,t + 3/2\,t^2 - 1/6\,t^3$ active2dplot(l0, x = -5 .. 10); 1

Laguerreplot2d1.eps active2d

The first few Laguerre polynomials are:

$$\begin{aligned}
l_0 &= 1\\
l_1 &= -t+1\\
l_2 &= \frac{1}{2}(t^2-4t+2)\\
l_3 &= \frac{1}{6}(-t^3+9t^2-18t+6)
\end{aligned}$$

2.3. **Differential Operators.**

Theorem 2.1. *Let*

$$\begin{aligned}
E_- &= tD^2+D\\
A_n &= tD^2+(1-t)D+n
\end{aligned}$$

with the Laguerre polynomial denoted as l_n*. Then*

$$E_- l_n = -nl_{n-1}$$

Proof. Let

$$E_- = A_n + tD - n$$

Multiply both sides of the equation by l_n.

$$\begin{aligned} E_{-}l_n &= [A_n + tD - n]l_n = -nl_{n-1} \\ tl_n' - nl_n &= -nl_{n-1} \\ -tl_n' + nl_n &= nl_{n-1} \end{aligned}$$

Then, we proceed in transform space. So, apply the Laplace transform to both sides of the equation with $L_n = \mathcal{L}\{l_n\} = \frac{(s-1)^n}{s^{n+1}}$ and simplify.

$$\begin{aligned} \mathcal{L}\{-tl_n' + nl_n\} &= -[-(sL_n(s) - l_n(0))]' + nL_n \\ &= -[-(L_n + sL_n')] + nL_n \\ &= L_n + sL_n' + nL_n \\ &= sL_n' + (1+n)L_n \\ &= s[\frac{(s-1)^n}{s^{n+1}}]' + (1+n)\frac{(s-1)^n}{s^{n+1}} \\ &= s[\frac{n(s-1)^{n-1}(s-1)^{n+1} - (n+1)s^n(s-1)^n}{s^{2n+2}}] \\ &\quad +(1+n)\frac{(s-1)^n}{s^{n+1}} \\ &= \frac{n(s-1)^{n-1}s^2 - (n+1)(s-1)^n s}{s^{n+2}} + (1+n)\frac{(s-1)^n s}{s^{n+2}} \\ &= \frac{n(s-1)^{n-1}}{s^{n-1}} \\ &= n\mathcal{L}\{l_{n-1}\}(s) \end{aligned}$$

□

Theorem 2.2. *Let*

$$\begin{aligned} E_O &= 2tD^2 + (2-2t)D - 1 \\ A_n &= tD^2 + (1-t)D + n \\ A_n l_n &= 0 \end{aligned}$$

with the Laguerre polynomial denoted as l_n. *Then*

$$E_O l_n = -(2n+1)l_n$$

Proof. Multiply A_n by 2

$$\begin{aligned} 2A_n &= 2tD^2 + 2(1-t)D + 2n \\ 2A_n l_n &= [2tD^2 + 2(1-t)D + 2n]l_n \\ 0 &= [2tD^2 + 2(1-t)D + 2n]l_n \end{aligned}$$

Add $-(2n+1)l_n$ to both sides of the equation and simplify

$$\begin{aligned}
-(2n+1)l_n &= [2tD^2 + 2(1-t)D + 2n]l_n - (2n+1)l_n \\
-(2n+1)l_n &= [2tD^2 + 2(1-t)D - 1]l_n \\
-(2n+1)l_n &= E_O l_n
\end{aligned}$$

□

Theorem 2.3. *Let*

$$\begin{aligned}
E_+ &= tD^2 + (1-2t)D + (t-1) \\
A_n &= tD^2 + (1-t)D + n \\
A_n l_n &= 0
\end{aligned}$$

Then

$$E_+ l_n = -(n+1)l_{n+1}$$

Proof. Let

$$E_+ = A_n - tD - (1-t-n)$$

Multiply by l_n

$$\begin{aligned}
E_+ l_n &= [A_n - tD - (1-t-n)]l_n = -(n+1)l_{n+1} \\
-tl_n' - (1-t+n)l_n &= -(n+1)l_{n+1} \\
tl_n' + (1-t+n)l_n &= (n+1)l_{n+1}
\end{aligned}$$

Then, we proceed in transform space. So, apply the Laplace transform to both sides of the equation. Let $L_n = \mathcal{L}\{l_n\} = \frac{(s-1)^n}{s^{n+1}}$

$$\begin{aligned}
\mathcal{L}\{tl_n' + (1-t+n)l_n\} &= -(sL_n(s) - l_n(0))' + (1+n)L_n(s) + L_n'(s) \\
&= -(L_n + sL_n') + (1+n)L_n + L_n' \\
&= -(s-1)L_n' + nL_n \\
&= -(s-1)[\frac{(s-1)^n}{s^{n+1}}]' + n[\frac{(s-1)^n}{s^{n+1}}] \\
&= -(s-1)[\frac{n(s-1)^{n-1}(s-1)^{n+1} - (n+1)s^n(s-1)^n}{s^{2n+2}}] \\
&\quad +n[\frac{(s-1)^n}{s^{n+1}}] \\
&= -(s-1)[\frac{n(s-1)^{n-1}(s-1)s - (n+1)(s-1)^n}{s^{n+2}}] \\
&\quad +n[\frac{(s-1)^n}{s^{n+1}}] \\
&= -(s-1)[\frac{(s-1)^{n-1}[ns-(n+1)(s-1)]}{s^{n+2}}] + \frac{n(s-1)^n}{s^{n+1}} \\
&= \frac{-(s-1)^n(n-s+1)}{s^{n+2}} + \frac{n(s-1)^n}{s^{n+1}} \\
&= \frac{(s-1)^n(-n+s-1+ns)}{s^{n+2}} \\
&= \frac{(s-1)^n(n+1)(s-1)}{s^{n+2}} \\
&= (n+1)\frac{(s-1)^{n+1}}{s^{n+2}} \\
&= (n+1)\mathcal{L}\{l_{n+1}\}(s)
\end{aligned}$$

□

2.4. **Lie Bracket.**

Theorem 2.4. *The Lie Bracket is defined as* $[A,B] = AB - BA$. *Let*

$$\begin{aligned}
E_O &= 2tD^2 + (2-2t)D - 1 \\
E_+ &= tD^2 + (1-2t)D + (t-1)
\end{aligned}$$

Then

$$[E_0, E_+] = -2E_0$$

Proof. Let

$$
\begin{aligned}
(2.5)\qquad [E_0, E_+] &= -2E_0 = E_0E_+ - E_+E_0 \\
(2.6)\qquad E_0E_+ &= 2tE_+^{''} + (2-2t)E_+^{'} - E_+ \\
(2.7)\qquad E_+E_0 &= tE_0^{''} + (1-2t)E_0^{'} + (t-1)E_0
\end{aligned}
$$

and

$$(2.8)\qquad -2E_+ = -2tD^2 + (-2+4t)D + (-2t+2)$$

We must find the first and second derivatives of E_0

$$
\begin{aligned}
E_0 &= 2tD^2 + (2-2t)D - 1 \\
E_0^{'} &= 2tD^3 + (4-2t)D^2 - 3D \\
E_0^{''} &= 2tD^4 + (6-2t)D^3 - 5D^2
\end{aligned}
$$

Now, we must find the first and second derivatives of E_+

$$E_+ = tD^2 + (1-2t)D + t - 1$$

Note that in the operator, $t-1$ means $(t-1)f(t)$, so we have

$$\frac{d}{dt}[t-1] = \frac{d}{dt}[(t-1)f(t)] = (t-1)f^{'}(t) + f(t) = (t-1)D + 1$$

Using that we get

$$E_+^{'} = tD^3 + (2-2t)D^2 + (t-3)D + 1$$

and

$$E_+^{''} = tD^4 + (3-2t)D^3 + (t-5)D^2 + 2D$$

Now, we substitute in the derivatives into (6) and (7)

$$
\begin{aligned}
E_0E_+ &= 2tE_+^{''} + (2-2t)E_+^{'} - E_+ \\
&= 2t[tD^4 + (3-2t)D^3 + (t-5)D^2 + 2D] + (2-2t)[tD^3 + (2-2t)D^2 + (t-3)D + 1] \\
&\quad -[tD^2 + (1-2t)D + t - 1] \\
&= 2t^2D^4 + D^3[6t - 4t^2 + 2t - 2t^2] + D^2[2t^2 - 10t + 4 - 8t + 4t^2 - t] \\
&\quad +D[4t + (2-2t)(t-3) + 2t - 1] + 2 - 2t - t + 1 \\
&= 2t^2D^4 + (-6t^2 + 8t)D^3 + (6t^2 - 19t + 4)D^2 + (-2t^2 + 14t - 7)D - 3t + 3
\end{aligned}
$$

$$
\begin{aligned}
E_+E_0 &= tE_0^{''} + (1-2t)E_0^{'} + (t-1)E_0 \\
&= t[2tD^4 + (6-2t)D^3 - 5D^2] + (1-2t)[2tD^3 + (4-2t)D^2 - 3D] \\
&\quad +(t-1)[2tD^2 + (2-2t)D - 1] \\
&= 2t^2D^4 + D^3[6t - 2t^2 + 2t - 4t^2] + D^2[-5t + 4 - 10t + 4t^2 + 2t^2 - 2t] \\
&\quad +D[6t - 3 - 2t^2 + 4t - 2] - t + 1 \\
&= 2t^2D^4 + (-6t^2 + 8t)D^3 + (6t^2 - 17t + 4)D^2 + (-2t^2 + 10t - 5)D - t + 1
\end{aligned}
$$

Now, we plug everything into (5)

$$
\begin{aligned}
E_0E_+ - E_+E_0 &= (2t^2 - 2t^2)D^4 + [(-6t^2 + 8t) - (-6t^2 + 8t)]D^3 + [(6t^2 - 19t + 4) \\
&\quad -(6t^2 - 17t + 4)]D^2 + [(-2t^2 + 14t - 7) - (-2t^2 + 10t - 5)]D \\
&\quad -3t + 3 - (-t + 1) \\
&= -2tD^2 + (4t - 2)D - 2t + 2
\end{aligned}
$$

Then using (5) and (8) we conclude that

$$
[E_0, E_+] = -2E_+
$$

$\square$

2.5. **Properties of the Laguerre Function.**

Theorem 2.5. *Let*

$$
\mathcal{L}\{l_n(at)\} = \sum_{k=0}^{n} \tbinom{n}{k}(-1)^k a^k \mathcal{L}\{\frac{t^k}{k!}\}
$$

Then

$$
\mathcal{L}\{l_n(at)\} = \frac{(s-a)^n}{s^{n+1}} \quad when \quad a \in \Re
$$

Proof. We take the Laplace transform of the summation and simplify

$$
\begin{aligned}
\mathcal{L}\{l_n(at)\} &= \sum_{k=0}^{n} \tbinom{n}{k}(-1)^k a^k \mathcal{L}\{\frac{t^k}{k!}\} \\
&= \sum_{k=0}^{n} \tbinom{n}{k}(-1)^k a^k \frac{1}{s^{k+1}} \\
&= \frac{1}{s^{n+1}} \sum_{k=0}^{n} \tbinom{n}{k}(-a)^k s^{n-k}
\end{aligned}
$$

following from the binomial theorem,

$$
(s+a)^n = \sum_{k=0}^{n} \tbinom{n}{k} a^k s^{n-k}
$$

we get

$$(s-a)^n = \sum_{k=0}^{n} \binom{n}{k} (-a)^k s^{n-k}$$

So,

$$\mathcal{L}\{l_n(at)\} = \frac{1}{s^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-a)^k s^{n-k} = \frac{(s-a)^n}{s^{n+1}}$$

$\square$

Theorem 2.6.

$$\sum_{k=0}^{n} \binom{n}{k} a^k l_k(t)(1-a)^{n-k} = l_n(at)$$

Proof. Apply the Laplace transform.

$$\begin{aligned}
\mathcal{L}\{\sum_{k=0}^{n} \binom{n}{k} a^k l_k(t)(1-a)^{n-k}\} &= \mathcal{L}\{l_n(at)\} \\
\mathcal{L}\{\sum_{k=0}^{n} \binom{n}{k} a^k l_k(t)(1-a)^{n-k}\} &= \sum_{k=0}^{n} \binom{n}{k} a^k \frac{(s-1)^k}{s^{k+1}} (1-a)^{n-k} \\
&= \frac{1}{s} \sum_{k=0}^{n} \binom{n}{k} a^k \frac{(s-1)^k}{s^k} (1-a)^{n-k} \\
&= \frac{1}{s} \sum_{k=0}^{n} \binom{n}{k} (a - \frac{a}{s})^k (1-a)^{n-k}
\end{aligned}$$

using the binomial theorem, we get

$$\begin{aligned}
\frac{1}{s} \sum_{k=0}^{n} \binom{n}{k} (a - \frac{a}{s})^k (1-a)^{n-k} &= \frac{1}{s}(1 - a + a - \frac{a}{s})^n \\
&= \frac{1}{s}(1 - \frac{a}{s})^n \\
&= \frac{1}{s}(\frac{s-a}{s})^n \\
&= \frac{(s-a)^n}{s^{n+1}} \\
\mathcal{L}\{\sum_{k=0}^{n} \binom{n}{k} a^k l_k(t)(1-a)^{n-k}\} &= \mathcal{L}\{l_n(at)\}
\end{aligned}$$

By taking the inverse Laplace transform, we can conclude that

$$\sum_{k=0}^{n} \tbinom{n}{k} a^k l_k(t)(1-a)^{n-k} = l_n(at)$$

□

Theorem 2.7. $\int_0^t l_n(x)dx = l_n(t) - l_{n+1}(t)$

Proof. We know that $l_n * 1(t) = \int_0^t l_n(x)dx$

By the convolution theorem, we also know that

$$\mathcal{L}\{l_n * 1\}(s) = \mathcal{L}\{l_n\}\mathcal{L}\{1\}$$

So we take the Laplace transform of l_n and 1

$$\begin{aligned}
\mathcal{L}\{l_n\}\mathcal{L}\{1\} &= \frac{1}{s}\frac{(s-1)^n}{s^{n+1}} \\
\mathcal{L}\{l_n\}\mathcal{L}\{1\} &= (1 - \frac{s-1}{s})\frac{(s-1)^n}{s^{n+1}}
\end{aligned}$$

Since

$$\mathcal{L}\{l_n * 1\}(s) = \mathcal{L}\{l_n\}\mathcal{L}\{1\}$$

we have

$$\begin{aligned}
\mathcal{L}\{l_n * 1\}(s) &= \frac{(s-1)^n}{s^{n+1}} - \frac{(s-1)^{n+1}}{s^{n+2}} \\
\mathcal{L}\{l_n * 1\}(s) &= \mathcal{L}\{l_n\}(s) - \mathcal{L}\{l_{n+1}\}(s)
\end{aligned}$$

Apply the inverse Laplace transform

$$l_n * 1 = l_n(t) - l_{n+1}(t)$$

By convolution, we can conlude that

$$\int_0^t l_n(x)dx = l_n(t) - l_{n+1}(t)$$

□

Theorem 2.8. $\int_0^t l_n(x)l_m(t-x)dx = l_{m+n}(t) - l_{m+n+1}(t)$

Proof. We know that $l_n * l_m(t) = \int_0^t l_n(x)l_m(t-x)dx$

By the convolution theorem, we also know that

$$\mathcal{L}\{l_n * l_m\}(s) = \mathcal{L}\{l_n\}\mathcal{L}\{l_m\}$$

So we take the Laplace transform of l_n and l_m

$$\begin{aligned}
\mathcal{L}\{l_n\}\mathcal{L}\{l_m\} &= \frac{(s-1)^n}{s^{n+1}}\frac{(s-1)^m}{s^{m+1}} \\
\mathcal{L}\{l_n\}\mathcal{L}\{l_m\} &= (1 - \frac{s-1}{s})\frac{(s-1)^{n+m}}{s^{n+m+1}}
\end{aligned}$$

Since

$$\mathcal{L}\{l_n * l_m\}(s) = \mathcal{L}\{l_n\}\mathcal{L}\{l_m\}$$

we have

$$\begin{aligned}\mathcal{L}\{l_n * l_m\}(s) &= \frac{(s-1)^{n+m}}{s^{n+m+1}} - \frac{(s-1)^{m+n+1}}{s^{m+n+2}} \\ \mathcal{L}\{l_n * l_m\}(s) &= \mathcal{L}\{l_{n+m}\}(s) - \mathcal{L}\{l_{n+m+1}\}(s)\end{aligned}$$

Apply the inverse Laplace transform

$$l_n * l_m = l_{m+n}(t) - l_{m+n+1}(t)$$

By convolution, we can conlude that

$$\int_0^t l_n(x) l_m(t-x)dx = l_{m+n}(t) - l_{m+n+1}(t)$$

$\square$

Theorem 2.9. *Recursion formula.*

$$l_{n+1}(t) = \frac{1}{n+1}[(2n+1-t)l_n(t) - nl_{n-1}(t)]$$

Proof. Apply the Laplace transform to both sides of the equation and simplify. Let $L_n = \mathcal{L}\{l_n\} = \frac{(s-1)^n}{s^{n+1}}$

$$\begin{aligned}
\mathcal{L}\{l_{n+1}(t)\} &= \mathcal{L}\{\frac{1}{n+1}[(2n+1-t)l_n(t) - nl_{n-1}(t)]\} \\
&= \frac{1}{n+1}[(2n+1)L_n + L_n' - nL_{n-1}] \\
&= \frac{1}{n+1}[(2n+1)[\frac{(s-1)^n}{s^{n+1}}] + [\frac{(s-1)^n}{s^{n+1}}]' - n[\frac{(s-1)^{n-1}}{s^n}]] \\
&= \frac{1}{n+1}[(2n+1)\frac{(s-1)^n}{s^{n+1}} + \frac{n(s-1)^{n-1}(s)^{n+1} - (n+1)s^n(s-1)^n}{s^{2n+2}} \\
&\quad - \frac{n(s-1)^{n-1}}{s^n}] \\
&= \frac{1}{n+1}[(2n+1)\frac{(s-1)^n}{s^{n+1}} + \frac{n(s-1)^{n-1}s - (n+1)(s-1)^n}{s^{n+2}} - \frac{n(s-1)^{n-1}}{s^n}] \\
&= \frac{1}{n+1}[(2n+1)\frac{(s-1)^{n-1}(s(s-1))}{s^{n+2}} + \frac{n(s-1)^{n-1}s - (n+1)(s-1)^n}{s^{n+2}} \\
&\quad - \frac{ns^2(s-1)^{n-1}}{s^{n+2}}] \\
&= \frac{1}{n+1}(\frac{(s-1)^{n-1}}{s^{n+2}})[(2n+1)(s(s-1)) + (n-s+1) - ns^2] \\
&= \frac{1}{n+1}(\frac{(s-1)^{n-1}}{s^{n+2}})[(n+1)(s-1)^2] \\
&= \frac{(s-1)^{n-1}}{s^{n+2}}(s-1)^2 \\
&= \frac{(s-1)^{n+1}}{s^{n+2}} \\
&= \mathcal{L}\{l_{n+1}(t)\}
\end{aligned}$$

By applying the inverse Laplace transform we can conclude that

$$l_{n+1}(t) = \frac{1}{n+1}[(2n+1-t)l_n(t) - nl_{n-1}(t)]$$

□

3. Using Differential Equations to Compute Laplace Transforms

3.1. **Introduction.** The Laplace transform of a function, F(t), is

$$f(s) = \int_0^\infty e^{-at}F(t)dt.$$

Theorem **??** and Table 1 are based on this definition.

Theorem 3.1. *If $f(s) = L\{F(t)\}$, then*

(1) $sf(s) - F(0) = L\{F'(t)\}$
(2) $f'(s) = L\{-tF(t)\}$

Therefore, the following table is created.

$f(s)$	$F(t)$
x	X
$sx - X(0)$	X'
$s^2x - sX(0) - X'(0)$	X''
$-x'$	tX
$-sx' - x$	tX'
$-s^2x' - 2sx + X(0)$	tX''
x''	t^2X
$sx'' + 2x'$	t^2X'
$s^2x'' + 4sx' + 2x$	t^2X''

TABLE 1. Basic Laplace Transforms

There are several other theorems that will also be useful to solve Laplace Transforms using differential equations.

Theorem 3.2. *If $X(t) \sim At^{\alpha}(\alpha > -1)$ as $t \to 0$,*

$$x(s) \sim \frac{A\Gamma(\alpha+1)}{s^{\alpha+1}} \quad as \quad s \to \infty$$

Theorem 3.3. *If $X(t) \sim Bt^{\beta}(\beta > -1)$ as $t \to \infty$,*

$$x(s) \sim \frac{B\Gamma(\beta+1)}{s^{\beta+1}} \quad as \quad s \to 0$$

3.2. **Method 1.** The first method used to compute Laplace Transforms uses Table **??** above to find the proper differential equations.

Theorem 3.4. *The Laplace Transform of the zeroth-order Bessel Function is the following:*

$$L\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}$$

and the Laplace Transform of the first-order Bessel Function is:

$$L\{J_1(t)\} = \frac{\sqrt{s^2+1} - s}{\sqrt{s^2+1}}$$

Proof of Theorem **??**. The Laplace Transform of the zeroth-order Bessel Function, denoted $J_0(t)$, is the solution of the differential equation $tX'' + X' + tX = 0$. By applying Table **??** to get the Laplace Transform, one can derive the equation $s^2x' - 2sx + X(0) + sx - X(0) - x' = 0$. Putting this into the standard form of the differential equation gives:

$$x' + \frac{s}{s^2+1}x = 0.$$

In order to solve this differential equation, one has to find the Integrating factor, which is defined by

$$I(t) = e^{\int \frac{s}{s^2+1} ds}$$

By solving the integral and simplifying, the integrating factor is shown to be $\sqrt{s^2+1}$. Multiplying the differential equation by the integrating factor gives $\sqrt{s^2+1}\, x' + \sqrt{s^2+1}\frac{s}{s^2+1}x = 0$, and integrating both sides of the equation gives $\sqrt{s^2+1}\, x = c$, where c is the integration constant. From this we can see that $x(s) = \frac{c}{\sqrt{s^2+1}}$. According to Theorem **??**, $c = 1$ because $x(s) \sim 1/s$ as $s \to \infty$. Therefore,

$$L\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}.$$

In order to derive $L\{J_1(t)\}$, the equality $J_0'(t) = -J_1(t)$ and Theorem **??** are used by placing $F(t) = J_0(t)$ and $F'(t) = -J_1(t)$. Then

$$L\{J_1(t)\} = \frac{\sqrt{s^2+1} - s}{\sqrt{s^2+1}}$$

and the Theorem is proved.

□

Theorem 3.5. *The Laplace Transform of the function $X(t) = \sin a\sqrt{t}$ can be stated as:*

$$L\{\sin a\sqrt{t}\} = \frac{a\sqrt{\pi}}{2s^{3/2}}e^{\frac{-a^2}{4s}}$$

Proof of Theorem **??**. In order to prove this theorem, the equation $X(t) = \sin(a\sqrt{t})$ must be used. To find a differential equation to solve for $x(s)$ one must take the derivative twice and set the equation equal to zero to get the constants. Putting these 3 equations together to equal zero results in the differential equation $4tX'' + 2X' + a^2X = 0$. Using the table, one can then take the Laplace transform of this equation showing that $4(-s^2x' - 2sx + 4X(0)) + 2(sx - X(0)) + a^2x = 0$.

By simplifying and changing to the standard form of the differential equation, the result is

$$x' + \frac{6s - a^2}{4s^2}x = -2X(0).$$

Multiplying by the integrating factor of $e^{\frac{a^2}{4s}}s^{3/2}$ shows that the equation is the product rule of $(e^{\frac{a^2}{4s}}s^{3/2}x)' = 0$. Integrating and dividing gives

$$x(s) = \frac{ce^{\frac{-a^2}{4s}}}{s^{3/2}}$$

where c is the integrating constant. One can see that $X(t) \sim a\sqrt{t}$ when $t \to 0$, so according to Theorem **??** $x(s) \sim \frac{a\sqrt{\pi}}{2s^{3/2}}$ when $s \to \infty$ because $\Gamma(3/2) = \frac{\sqrt{\pi}}{2}$. Therefore, $c = \frac{a\sqrt{\pi}}{2}$ because x(s) is also similar to $\frac{c}{s^{3/2}}$. When c is plugged in c, the final result is

$$L\{\sin a\sqrt{t}\} = \frac{a\sqrt{\pi}}{2s^{3/2}}e^{\frac{-a^2}{4s}}.$$

□

Theorem 3.6. *The Laplace Transform of the zeroth-order Bessel Function with $a\sqrt{t}$ plugged in is shown as:*

$$L\{J_0(a\sqrt{t})\} = \frac{e^{\frac{-a^2}{4s}}}{s}$$

Proof of Theorem **??**. By taking the zeroth-order Bessel Function and plugging in $a\sqrt{t}$, one gets that $X(t) = J_0(a\sqrt{t})$, which can be solved by the differential equation $4tX'' + 4X' + a^2X = 0$. Taking the Laplace transform of this equation gives $4(-s^2x' - 2sx + X(0)) + 4(sx - X(0)) + a^2x = 0$. Through simplification and division, the standard form of the differential equation is the following:

$$x' + \frac{4s - a^2}{4s^2}x = 0.$$

The simplified integrating factor for this equation is $I = e^{\frac{a^2}{4s}}s$. By multiplying the integrating factor by the differential equation, recognizing the product rule, integrating, and solving for x, the result is

$$x(s) = \frac{ce^{\frac{-a^2}{4s}}}{s}$$

According to Theorem **??**, since $X(t) \sim 1$ as $t \to 0$, then $x(s) \sim 1/s$ as $s \to \infty$. Therefore, $c = 1$ and

$$L\{J_0(a\sqrt{t})\} = \frac{e^{\frac{-a^2}{4s}}}{s}.$$

□

3.3. **Method 2.** This new method requires another table to be constructed.

$f(s)$	$F(t)$
x	X
x'	$-tX$
x''	t^2X
$sx - X_0$	X'
sx'	$-tX' - X$
sx''	$t^2X' + 2tX$
$s^2x - sX_0 - X_1$	X''
$s^2x' + X_0$	$-tX'' - 2X'$
s^2x''	$t^2X'' + 4tX' + 2X$

TABLE 2. Inverse Laplace Transforms

Here $X_0 = \lim_{s\to\infty} sx(s)$ and $X_1 = \lim_{s\to\infty} (s^2x(s) - sX_0)$

Theorem 3.7. *Note the following equality:*

$$\frac{e^{-a\sqrt{s}}}{\sqrt{s}} = L\{\frac{e^{\frac{-a^2}{4t}}}{\sqrt{\pi t}}\}$$

Proof of Theorem **??**. In order to prove the equality true, one must begin with the equation

$$x(s) = \frac{e^{-a\sqrt{s}}}{\sqrt{s}}.$$

Then the equation will be differentiated twice and simplified to get the differential equation,

$$4sx'' + 6x' - a^2x = 0.$$

By taking the inverse Laplace transform using Table 2, the result is $4t^2X' + 8tX - 6tX - a^2X$, which can be simplified and put in the standard form of the differential equation to give:

$$X' + \frac{2t - a^2}{4t^2}X = 0.$$

The integrating factor of this equation is $\sqrt{t}e^{\frac{-a^2}{4t}}$. When this is multiplied by the equation and solved for $X(t)$, the result is

$$X(t) = \frac{ce^{\frac{-a^2}{4t}}}{\sqrt{t}}$$

According to Theorem **??**, since $X(t) \sim \frac{c}{\sqrt{t}}$ as $t \to \infty$, then $x(s) \sim \frac{c\sqrt{\pi}}{\sqrt{s}}$ as $s \to 0$ because $\Gamma(1/2) = \sqrt{\pi}$. Then $c = \frac{1}{\sqrt{\pi}}$, because $x(s)$ is also similar to $\frac{1}{\sqrt{s}}$ as $s \to 0$. By plugging in the value of c to the expression of X(t), the result is the following:

$$\frac{e^{-a\sqrt{s}}}{\sqrt{s}} = L\{\frac{e^{\frac{-a^2}{4t}}}{\sqrt{\pi t}}\},$$

thus proving the theorem.

□

Theorem 3.8. *The Bessel Differential Equation of the n^{th} order can be defined as*

$$t^2X'' + tX' + (t^2 - n^2)X = 0$$

where the solutions of this differential equation are the n^{th} order Bessel Functions.

Proof of Theorem **??**. From the end of Example 1 comes the equation

$$x(s) = \frac{(\sqrt{s^2+1} - s)^n}{\sqrt{s^2+1}}$$

. In order to solve for the original equation, one must take the first and second derivative and combine to form a differential equation. The first derivative is $x'(s) =$ and the second derivative is $x''(s) =$. The resulting differential equation is

$$(s^2+1)x'' + 3sx' + (1 - n^2)x = 0$$

To take the inverse Laplace transform of this equation, one can use the Table **??** to get

$$t^2X'' + tX' + (t^2 - n^2)X = 0$$

This equation is the Differential Bessel's Equation of n^{th} order. Therefore, using Theorem **??**, X(t) must equal $J_n(t)$.

□

Theorem 3.9. *The following Laplace Transform can be shown:*

$$\frac{e^{s-\sqrt{s^2+1}}}{\sqrt{s^2+1}} = L\{J_0(\sqrt{t^2+2t})\}.$$

This example displays a more difficult problem to solve and requires supplementing Table 2 with additional entries.

$f(s)$	$F(t)$
x'''	$-t^3X$
s^2x'''	$-t^3X'' - 6t^2X' - 6tX$

TABLE 3. Additional Inverse Laplace Transforms

Proof of Theorem **??**. To prove Theorem **??**, one starts with the equation

$$x(s) = \frac{e^{s-\sqrt{s^2+1}}}{\sqrt{s^2+1}}.$$

The equation is then differentiated three times and the equations are combined those equations to find the differential equation:

$$(s^2+1)x''' - (3s^2-5s+3)x'' + (2s^2-10s+7)x' + (2s-5)x = 0.$$

From Table 2 and Table 3, the inverse Laplace Transform of the equation is $(-t^3-3t^2-2t)X''+(-t^2-12t-2)X'+(-t^3-3t^2-3t-1)X = 0$, which simplifies to

$$(t+1)(t^2+2t)X'' + (t^2+2t+2)X' + (t+1)^3X = 0.$$

When a variable, y, is set equal to t^2+2t, then this substitution can be plugged into the equation to get $4yX''(y) + 4X'(y) + X(y) = 0$. By observation this differential equation is a form of the equation from Theorem **??** where

$$L\{J_0(a\sqrt{t})\} = \frac{e^{\frac{-a^2}{4s}}}{s},$$

which solves the differential equation, $4tX''+4X'+a^2X = 0$. Then by substituting y for t, this shows

$$\frac{e^{s-\sqrt{s^2+1}}}{\sqrt{s^2+1}} = L\{J_0(\sqrt{t^2+2t})\}.$$

□

These theorems demonstrate how to calculate Laplace transforms by using differential equations. The properties of the Laplace Transform shown in the tables and solving the associated differential equations makes calculating Laplace transforms much easier.

4. Laplace Transform and Linear Recursion Relations

Abstract. In this part of the project we will be talking about how to use the Laplace transform to solve linear recursion relations. More specifically, we are looking at linear recursion relations of order 2 which have the form:

$$a_{n+2} + ba_{n+1} + ca_n = f(n),$$

where b and c are real numbers and $f(n)$ is a known sequence. Our goal is to find a closed formula for a_n. We will see how the Laplace transform could help us in solving these relations.

4.1. **Introduction.** In order to solve the linear recursion relations order 2: $a_{n+2} + ba_{n+1} + ca_n = f(n)$, we let $y(t) = a_n$, and $f(t) = f(n)$ for $n \leq t < n+1$ where $n = 0, 1, 2, \ldots$. Now the relation becomes:

$$y(t+2) + by(t+1) + cy(t) = f(t)$$

Now take the Laplace transform of both sides, we have:

$$\mathcal{L}\{y(t+2)\} + b\mathcal{L}\{y(t+1)\} + c\mathcal{L}\{y(t)\} \quad = \quad \mathcal{L}\{f(t)\} \tag{4.1}$$

In later sections, we will learn how to simplify the left side of Equation (**??**) in terms of $Y(s) = \mathcal{L}\{y(t)\}$, and also how to compute the Laplace transform of the right side, sometimes called the **forcing function**. After we have the closed form of $Y(s)$, we will see how we could compute the Laplace inverse of that closed form to get the closed form of the sequence $\{a_n\}$.

4.2. Basic Laplace Transform Formulas.

4.2.1. *Dealing with the left side.* We will need the following proposition to simplify the left side of Equation (**??**):

Proposition 4.1. *With notation as above we have*

$$\mathcal{L}\{y(t+1)\} \quad = \quad e^s Y(s) - \frac{a_0 e^s(1-e^{-s})}{s} \tag{4.2}$$

$$\mathcal{L}\{y(t+2)\} \quad = \quad e^{2s} Y(s) - \frac{e^s(1-e^{-s})(a_0 e^s + a_1)}{s} \tag{4.3}$$

with $Y(s) = \mathcal{L}\{y(t)\}$.

Proof. [1] Let $u = t+1$, we have:

$$\begin{aligned}
\mathcal{L}\{y(t+1)\} &= \int_0^\infty e^{-st} y(t+1)dt \\
&= e^s \int_1^\infty e^{-su} y(u)du \\
&= e^s \int_0^\infty e^{-su} y(u)du - e^s \int_0^1 e^{-su} y(u)du \\
&= e^s Y(s) - e^s \int_0^1 e^{-su} a_0 du \\
&= e^s Y(s) - \frac{a_0 e^s (1-e^{-s})}{s}
\end{aligned}$$

using the fact that $Y(t) = a_0$ for $0 \le t < 1$. This proves Equation (**??**).

Let $u = t+2$, we have:

$$\begin{aligned}
\mathcal{L}\{y(t+2)\} &= \int_0^\infty e^{-st} y(t+2)dt \\
&= e^{2s} \int_2^\infty e^{-su} y(u)du \\
&= e^{2s} \left[\int_0^\infty e^{-su} y(u)du - \int_0^1 e^{-su} y(u)du \right. \\
&\quad \left. - \int_1^2 e^{-su} y(u)du \right] \\
&= e^{2s} Y(s) - e^{2s} \int_0^1 e^{-su} a_0 du - e^{2s} \int_1^2 e^{-su} a_1 du \\
&= e^{2s} Y(s) - \frac{a_0 e^{2s}(1-e^{-s})}{s} - \frac{a_1 e^{2s}(e^{-s} - e^{-2s})}{s} \\
&= e^{2s} Y(s) - \frac{e^s(1-e^{-s})(a_0 e^s + a_1)}{s}
\end{aligned}$$

This proves Equation (**??**). □

Now we have successfully expressed $\mathcal{L}\{y(t+2)\}$ and $\mathcal{L}\{y(t+1)\}$ in terms of $Y(s) = \mathcal{L}\{y(t)\}$, so we could express the left side of (1) in form of $Y(s)$.

4.2.2. *Dealing with the right side.* Next let's see how we can deal with the right side of Equation (**??**), finding the $\mathcal{L}\{f(t)\}$. To do this, we

[1] These 2 identities and their proofs are taken from [**?**].

need to use the Heaviside function $h_c(t)$ and on-off switch $\chi_{[a,b)}$ that are defined as followed:

$$\begin{aligned} h_c(t) &= \begin{cases} 0 \text{ if } 0 \le t < c, \\ 1 \text{ if } c \le t. \end{cases} \\ \chi_{[a,b)} &= \begin{cases} 1 \text{ if } a \le t < b \ , \\ 0, \text{ elsewhere.} \end{cases} \end{aligned}$$

It is easy to prove that $\chi_{[a,b)} = h_a - h_b$. Also we will need the following formula to compute the Laplace transform of our functions:

$$\mathcal{L}\{h_c(t)\} = \frac{e^{-sc}}{s}$$

This formula and also its proof can be found in [?].

Let's first express $f(t)$ in terms of Heaviside functions.

$$\begin{aligned} f(t) &= \begin{cases} f(0) \ , \text{ if } 0 \le t < 1 \ , \\ f(1) \ , \text{ if } 1 \le t < 2, \\ \vdots \\ f(n) \ , \text{ if } n \le t < n+1, \\ \vdots \end{cases} \\ &= f(0)\chi_{[0,1)} + f(1)\chi_{[1,2)} + \ldots \\ &= f(0)\left(h_0 - h_1\right) + f(1)\left(h_1 - h_2\right) + \ldots \\ &= f(0) + h_1\left(f(1) - f(0)\right) + h_2\left(f(2) - f(1)\right) + \ldots \end{aligned}$$

Now we have enough information to compute the Laplace transform for $f(t)$.

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \frac{f(0)}{s} + \frac{\left(f(1) - f(0)\right)e^{-s}}{s} + \frac{\left(f(2) - f(1)\right)e^{-2s}}{s} + \ldots \\ &= \frac{1}{s}\left(f(0)(1 - e^{-s}) + e^{-s}f(1)(1 - e^{-s}) + \cdots\right) \\ &= \frac{1 - e^{-s}}{s} \cdot \sum_{k=0}^{\infty} e^{-sk} f(k) \end{aligned}$$

So we can write

$$\mathcal{L}\left\{f(t)\right\} = \frac{1 - e^{-s}}{s}\, G(s), \tag{4.4}$$

where $G(s) = \sum_{k=0}^{\infty} e^{-sk} f(k)$. Below are some formulas regarding some simple forcing functions.

Proposition 4.2.

$$\begin{aligned}
\mathcal{L}\{f(t)=k\} &= \frac{k}{s} \\
\mathcal{L}\{f(t)=r^n, n\le t<n+1\} &= \frac{1-e^{-s}}{s(1-re^{-s})} \qquad (4.5)\\
\mathcal{L}\{f(t)=n, n\le t<n+1\} &= \frac{e^{-s}}{s(1-e^{-s})}
\end{aligned}$$

Proof. From (**??**), we have:

$$G(s)=k\sum_{i=0}^{\infty}e^{-si}=\frac{k}{1-e^{-s}}$$

Of course we need $s>0$ to use that geometric series formula. Now according to Equation (**??**), we will have $\mathcal{L}\{f(t)=k\}=\frac{k}{s}$.

When $f(n)=r^n$, with r is a constant. We have:

$$G(s)=\sum_{k=0}^{\infty}e^{-sk}r^k=\sum_{k=0}^{\infty}(re^{-s})^k=\frac{1}{1-re^{-s}}$$

Again, we need to limit s such that: $|re^{-s}|<1$, so we could use the geometric series formula. Now according to (**??**), we have:

$$\mathcal{L}\{f(t)=r^n, n\le t<n+1\}=\frac{1-e^{-s}}{s(1-re^{-s})}$$

When $f(n)=n$:

$$\begin{aligned}
G(s) &= \sum_{k=0}^{\infty}ke^{-ks} \\
&= \sum_{k=0}^{\infty}-\frac{d}{ds}\left[e^{-ks}\right] \\
&= -\frac{d}{ds}\left[\sum_{k=0}^{\infty}e^{-ks}\right] \\
&= -\frac{d}{ds}\left[\frac{1}{1-e^{-s}}\right] \\
&= \frac{e^{-s}}{(1-e^{-s})^2}
\end{aligned}$$

Again, from (**??**) we will have:

$$\mathcal{L}\{f(t)=n, n\le t<n+1\}=\frac{e^{-s}}{s(1-e^{-s})}$$

$\square$

Other kinds of forcing functions $f(n)$ will be introduced later.

5. The Homogeneous Case

Here we want to consider a recursion relation of the form

$$a_{n+2} - ba_{n+1} + ca_n = 0,$$

where b and c are real numbers. Such an equation is called **homogenous**. To get an idea of the general result let's consider the following example. Let's look at a very famous sequence, the Fibonacci sequence, which can be rewritten as:

$$a_{n+2} - a_{n+1} - a_n = 0,$$

where the initial conditions are $a_0 = 0$ and $a_1 = 1$. After setting up the $y(t)$ and taking the Laplace transform we have:

$$Y(s) = \frac{e^s(1-e^{-s})}{s} \cdot \frac{1}{e^{2s} - e^s - 1}$$

Let $r = e^s$. Then we can write $e^{2s} - e^s - 1 = r^2 - r - 1 = (r-\alpha)(r-\beta)$, with $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Using partial fractions we have:

$$\begin{aligned} Y(s) &= \frac{e^s(1-e^{-s})}{s} \cdot \frac{1}{(e^s-\alpha)(e^s-\beta)} \\ &= \frac{e^s(1-e^{-s})}{s(\alpha-\beta)} \cdot \left(\frac{1}{e^s-\alpha} - \frac{1}{e^s-\beta}\right) \\ &= \frac{1-e^{-s}}{s(\alpha-\beta)} \cdot \left(\frac{1}{1-\alpha e^{-s}} - \frac{1}{1-\beta e^{-s}}\right) \end{aligned}$$

By Proposition **??**, we have

$$\begin{aligned} a_n &= \frac{1}{\alpha-\beta}(\alpha^n - \beta^n) \\ &= \frac{1}{\sqrt{5}} \cdot \left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]. \end{aligned}$$

With this example as our guide we obtain the following theorem.

Theorem 5.1. *The general solution of a linear recursion relation of order 2: $a_{n+2} + ba_{n+1} + ca_n = 0$ will be determined by its characteristic polynomial $x^2 + bx + c$ as following:*

1. *If $x^2 + bx + c = 0$ has 2 separate real roots (α, β), the general solution would have the form of:*

$$a_n = A\alpha^n + B\beta^n$$

2. *If $x^2+bx+c = 0$ has a double root α, the general solution would have the form of:*

$$a_n = (An + B)\alpha^n$$

3. *If $x^2 + bx + c = 0$ has 2 complex roots $(\alpha + i\beta, \alpha - i\beta)$, the general solution would have the form of:*

$$a_n = u^n(A\cos(n\theta) + B\sin(n\theta))$$

with $u = \sqrt{\alpha^2 + \beta^2}$, $\theta = \sin^{-1}\frac{\beta}{u}$, and A, B are real constants in all cases.

Proof. From the relation equation, we take the Laplace transform for both sides and then use our formulas (**??**) and (**??**), we have:

$$Y(s) = \frac{e^s(1 - e^{-s})}{s} \cdot \frac{k_1 e^s + k_2}{e^{2s} + be^s + c} \tag{5.1}$$

with $k_1 = a_0 + a_0 b$, $k_2 = a_1$. Now:

1. If $x^2 + bx + c = 0$ has 2 separate real roots (α, β), we could rewrite $Y(s)$ as:

$$Y(s) = \frac{1 - e^{-s}}{s} \cdot \frac{k_2 e^{-s} + k_1}{(1 - \alpha e^{-s})(1 - \beta e^{-s})}$$

Then using partial fraction will give us:

$$Y(s) = \frac{1 - e^{-s}}{s} \cdot \left(\frac{k_3}{1 - \alpha e^{-s}} + \frac{k_4}{1 - \beta e^{-s}}\right)$$

Now use formula (**??**) we will have:

$$a_n = k_3\alpha^n + k_4\beta^n$$

with k_3 and k_4 are 2 real constants.

2. If $x^2 + bx + c = 0$ has a double root α. We can rewrite $Y(s)$ as:

$$Y(s) = \frac{1 - e^{-s}}{s} \cdot \frac{k_2 e^{-s} + k_1}{(1 - \alpha e^{-s})^2}$$

Using partial fraction will give us:

$$Y(s) = \frac{1 - e^{-s}}{s} \cdot \left(\frac{k_5}{(1 - \alpha e^{-s})^2} + \frac{k_6}{1 - \alpha e^{-s}}\right)$$

Now we need to use (**??**) to compute the function $f(n)$ which corresponds to the Laplace function of $\frac{1-e^{-s}}{s} \cdot \frac{1}{(1-\alpha e^{-s})^2}$. We have:

$$\begin{aligned}
\frac{1}{(1-\alpha e^{-s})^2} &= \frac{1}{1-\alpha e^{-s}} \cdot \frac{1}{1-\alpha e^{-s}} \\
&= \left(\sum_{k=0}^{\infty} e^{-sk}\alpha^k\right) \cdot \left(\sum_{k=0}^{\infty} e^{-sk}\alpha^k\right) \\
&= \left(1+\alpha e^{-s}+\alpha^2 e^{-2s}+\ldots\right) \cdot \\
&\quad \left(1+\alpha e^{-s}+\alpha^2 e^{-2s}+\ldots\right) \\
&= 1+2\alpha e^{-s}+3\alpha^2 e^{-2s}+\ldots \\
&= \sum_{k=0}^{\infty} e^{-sk}(k+1)\alpha^k
\end{aligned}$$

Therefore: $f(n) = (n+1)\alpha^n$. Now using this to compute the Laplace transform of $Y(s)$ will give us the general solution in the form of:

$$a_n = (An+B)\alpha^n$$

3. If $x^2+bx+c=0$ has 2 complex roots $(\alpha+i\beta, \alpha-i\beta)$, with α and β are 2 real constants. From (**??**) we have:

$$Y(s) = \frac{1-e^{-s}}{s} \cdot \frac{k_2 e^{-s}+k_1}{(1-(\alpha+i\beta)e^{-s})\,(1-(\alpha-i\beta)e^{-s})}$$

Using partial fraction gives us:

$$Y(s) = \frac{1-e^{-s}}{s} \cdot \left(\frac{k_7}{1-(\alpha+i\beta)e^{-s}} + \frac{k_8}{1-(\alpha-i\beta)e^{-s}}\right)$$

with:

$$k_7 = \frac{k_1\beta-(k_2+\alpha k_1)i}{2\beta}$$

$$k_8 = \frac{k_1\beta+(k_2+\alpha k_1)i}{2\beta}$$

Now use the Laplace inverse, we will have:

$$a_n = k_7(\alpha+i\beta)^n + k_8(\alpha-i\beta)^n$$

Note that we could express $\alpha \pm i\beta$ as $\sqrt{\alpha^2+\beta^2}(\cos\theta \pm i\sin\theta)$ with $\theta = \sin^{-1}\frac{\beta}{\sqrt{\alpha^2+\beta^2}}$. Let's call $u = \sqrt{\alpha^2+\beta^2}$. Use the De

Moivre's formula we will have:

$$\begin{aligned} a_n &= k_7 u^n\left(\cos(n\theta) + i\sin(n\theta)\right) + k_8 u^n\left(\cos(n\theta) - i\sin(n\theta)\right) \\ &= (k_7+k_8)u^n\cos(n\theta) + iu^n\sin(n\theta)(k_7-k_8) \\ &= k_1u^n\cos(n\theta) + \frac{k_2+\alpha k_1}{\beta}\cdot u^n\sin(n\theta) \\ a_n &= u^n\left(A\cos(n\theta) + B\sin(n\theta)\right) \end{aligned}$$

□

5.1. **Non-homogenous Recursion Relations.** The following theorems are some theorems that gives us great insight into how to deal with non-homogeneous cases.

Theorem 5.2. *Let $a_{n,p}$ be a fixed particular solution to the second order linear recursion relation*

$$a_{n+2} + ba_{n+1} + ca_n = f(n).$$

Then any other solution has the form $a_n = a_{n,h} + a_{n,p}$, for some homogeneous solution $a_{n,h}$.

Proof. The theorem follows from linearity of the recursion relation equation. □

Theorem 5.3. *Let a_{n,p_1} and a_{n,p_2} be particular solutions of the following relations respectively:*

$$a_{n+2} + ba_{n+1} + ca_n = f_1(n)$$

$$a_{n+2} + ba_{n+1} + ca_n = f_2(n)$$

then a particular solution of the following relation

$$a_{n+2} + ba_{n+1} + ca_n = f(n)$$

with $f(n) = f_1(n) + f_2(n)$, will have the form of:

$$a_{n,p} = a_{n,p_1} + a_{n,p_2}$$

Proof. Again the theorem follows from linearity of the recursion relation equation. □

The biggest problem in solving a non-homogeneous case is to find the particular solution. This will be discussed later in this section after we know some more formulas. For now let's look at an example first.

5.2. **Example.** Find the closed form of a_n which is defined by the following relation:

$$a_{n+2} - 5a_{n+1} + 6a_n = 4^n, \quad a_0 = 0, a_1 = 1.$$

After taking the Laplace transform and simplifying, we have:

$$Y(s) = \frac{1-e^{-s}}{2s} \cdot \left(\frac{1}{1-4e^{-s}} - \frac{1}{1-2e^{-s}} \right)$$

Taking the Laplace inverse will give us:

$$a_n = \frac{1}{2}(4^n - 2^n)$$

Note that if we want to use our theorem for this example, our general homogeneous solution have the form of:

$$a_{n,h} = c_1 3^n + c_2 2^n$$

Later on we will know that our general particular solution in this case is:

$$a_{n,p} = c_3 4^n$$

So our general solution would be:

$$a_n = c_1 3^n + c_2 2^n + c_3 4^n$$

In this case, c_1 happens to be zero. However, if we solve this relation in the homogeneous case, we will have our solution to be: $a_n = 3^n - 2^n$. Now we can see that the existence of the forcing function does not only add more terms to the general solution but it also changes the coefficients of the terms in the homogeneous solution.

5.2.1. *Dealing with the forcing function.* Suppose $f(n)$ is a forcing function with Laplace transform of the form $F(s) = \frac{p_1(e^{-s})}{sq_1(e^{-s})}$, where p_1 and q_1 are polynomials. Here we will consider recursion relations of the form

$$a_{n+2} + ba_{n+1} + ca_n = f(n).$$

After taking the Laplace transform for both sides and simplifying, we have the following kind of equation:

$$(e^{2s} + be^s + c)Y(s) = \frac{p(e^{-s})}{sq(e^{-s})},$$

again, where p and q are polynomials. The characteristic polynomial can be factored into 2 (or it can be one) factors $(e^s - \alpha)$ and $(e^s - \beta)$, with α and β can be real or complex numbers. Also, since we do not limit our numbers in the real set, our polynomial $q(e^{-s})$ can also be factored into a product of terms in the form of $(1 - re^{-s})$. Note that the terms that we just mentioned are interchangeable. For example: $(e^s - \alpha) = e^s(1 - \alpha e^{-s})$. So we could rewrite our equation as:

$$Y(s) = \frac{1}{s} \cdot \frac{p(e^{-s})}{\prod(1 - r_k e^{-s})^p e^{-sq}}$$

Now we can use partial fraction to break it into simple fractions like:

$$Y(s) = \frac{1}{s} \cdot \sum \frac{c_i}{(1 - r_k e^{-s})^p}$$

One might wonder what if we have terms like $\frac{1}{e^{-s}}$? Actually the term e^{-s} does appear sometimes in the denominator but when we get to this stage, after we have everything in simple fractions type, we will never have the fraction $\frac{1}{e^{-s}}$, which is e^s. We can prove this easily by using the fact that $Y(s)$ needs to go to zero when s goes to infinity. While all terms $\frac{1}{s(1-re^{-s})^p}$ goes to zero when s goes to infinity, $\frac{e^s}{s}$ on the other hand goes to infinity. Therefore, $\lim_{s\to\infty} Y(s) \neq 0$. So, by contradiction, we have proved that we will never have the fraction $\frac{1}{e^{-s}}$ in our final step.

Now all we need to do is to find the general formula for those kind of fractions. Let $H(n,p)$ be a function such that:

$$\sum_{k=0}^{\infty} e^{-sk} H(k,p) = \frac{1}{(1 - re^{-s})^p}$$

We will use induction method to prove the following proposition:

Proposition 5.4.

$$H(n,p) = \frac{r^n}{(p-1)!} \cdot \prod_{k=1}^{p-1}(n+k) \tag{5.2}$$

Proof. When $p = 1$, we need to prove that $H(n,1) = r^n$. This is obviously true according to our formula (**??**). Now suppose (**??**) is true from 1 to p, we will need to prove that it's also true for $p+1$. We have:

$$\begin{aligned}
\sum_{k=0}^{\infty} e^{-sk} H(k,p+1) &= \frac{1}{(1-re^{-s})^{p+1}} \\
&= \frac{1}{e^{-s}p} \cdot \frac{d}{dr}\left[\frac{1}{(1-re^{-s})^{p}}\right] \\
&= \frac{1}{e^{-s}p} \cdot \frac{d}{dr}\left[\sum_{k=0}^{\infty} e^{-sk} H(k,p)\right] \\
&= \frac{1}{p} \cdot \frac{d}{dr}\left[\sum_{k=0}^{\infty} e^{-s(k-1)} H(k,p)\right] \\
&= \frac{1}{p} \cdot \sum_{k=0}^{\infty}\left(e^{-s(k-1)} \cdot \frac{d}{dr}\left[H(k,p)\right]\right)
\end{aligned}$$

Now look at the formula (**??**), we have $H(0,p)$ does not depend on the value of r. Therefore, its first derivative with respect to r is zero. So now in the summation, we just need k from 1 to ∞. But we want to match up with the left side of the equation, so we need to change it to $0 \to \infty$. After doing this, we will have:

$$\begin{aligned}
\sum_{k=0}^{\infty} e^{-sk} H(k,p+1) &= \frac{1}{p} \cdot \sum_{k=0}^{\infty} e^{-sk} \frac{d}{dr}\left[H(k+1,p)\right] \\
H(n,p+1) &= \frac{1}{p} \cdot \frac{d}{dr}\left[H(n+1,p)\right] \\
&= \frac{1}{p} \cdot \frac{d}{dr}\left[\frac{r^{n+1}}{(p-1)!} \cdot \prod_{k=1}^{p-1}(n+1+k)\right] \\
&= \frac{(n+1)r^{n}}{p!} \cdot \prod_{k=1}^{p-1}(n+1+k) \\
H(n,p+1) &= \frac{r^{n}}{p!} \cdot \prod_{k=1}^{p}(n+k)
\end{aligned}$$

This means our formula (**??**) is also true for $p+1$. By induction, we have proved that (**??**) is true for all $p > 0$. $\square$

Note that with the way we defined $H(n,p)$, $H(n,p)$ is not the Laplace inverse function of the fraction $\frac{1}{(1-re^{-s})^{p}}$. Instead:

$$H(n,p) = \mathcal{L}^{-1}\left\{\frac{1-e^{-s}}{s} \cdot \frac{1}{(1-re^{-s})^{p}}\right\} \tag{5.3}$$

Luckily, when we try to compute $Y(s)$, we usually get terms that have a common factor of $\frac{1-e^{-s}}{s}$. If we encounter some terms like $\frac{1}{s(1-re^{-s})^p}$ by itself, then we just need to multiply the numerator and denominator by $(1-e^{-s})$ and then use partial fraction to express $\frac{1}{(1-e^{-s})(1-re^{-s})^p}$ in terms of $\frac{1}{1-e^{-s}}$, and $\frac{1}{(1-re^{-s})^k}$, with $0 < k \leq p$. Finally, use the formula of $H(n,p)$ to compute the Laplace inverse of $Y(s)$. Now let's look at an example to see how we could use (**??**).

5.2.2. *Example.* Find the closed form of a_n which is defined by the following relation:

$$a_{n+2} - 5a_{n+1} + 6a_n = n2^n, \quad a_0 = 0, \quad a_1 = 1$$

First, let's calculate the Laplace transform of the right side. From (**??**), we have:

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \frac{1-e^{-s}}{s} \cdot \sum_{k=0}^{\infty} e^{-sk} k 2^k \\
&= \frac{1-e^{-s}}{s} \cdot \left(\sum_{k=0}^{\infty} e^{-sk}(k+1)2^k - \sum_{k=0}^{\infty} e^{-sk} 2^k \right) \\
&= \frac{1-e^{-s}}{s} \cdot \left(\sum_{k=0}^{\infty} e^{-sk} H(k,2) - \sum_{k=0}^{\infty} (2e^{-s})^k \right) \\
&= \frac{1-e^{-s}}{s} \cdot \left(\frac{1}{(1-2e^{-s})^2} - \frac{1}{1-2e^{-s}} \right)
\end{aligned}$$

Now take the Laplace transform for both sides of the relation equation and use our known formulas, we have:

$$\begin{aligned}
Y(s)(e^{2s} - 5e^s + 6) &= \frac{1-e^{-s}}{s} \cdot \left(\frac{1}{(1-2e^{-s})^2} - \frac{1}{1-2e^{-s}} + e^s \right) \\
Y(s) &= \frac{1-e^{-s}}{s} \cdot \\
&\quad \left(\frac{3}{1-3e^{-s}} - \frac{5}{2(1-2e^{-s})} - \frac{1}{2(1-2e^{-s})^3} \right) \\
a_n &= 3^{n+1} - \frac{5}{2} 2^n - \frac{1}{2} H(n,3)\{r=2\} \\
a_n &= 3^{n+1} - 5.2^{n-1} - \frac{1}{4}(n+1)(n+2)2^n \\
a_n &= 3^{n+1} - (n^2 + 3n + 12).2^{n-2}
\end{aligned}$$

We can easily check the result and see that this is a correct formula for a_n. And now we have enough formulas to derive the following theorem about particular solution.

Theorem 5.5. *The particular solution of the second order linear recursion relation* $a_{n+2}+ba_{n+1}+ca_n = g(n)r^n$*, in which* $g(n)$ *is a polynomial degree* k *of* n*, will be determined as followed:*

1. *If the constant* r *does not match any roots (either real or complex) of the equation* $x^2+bx+c=0$*, then the particular solution would be in form of:*
$$a_{n,p} = g^*(n)r^n$$
with $g^*(n)$ *is a polynomial degree* k *of* n*.*
2. *If the constant* r *matches one of the 2 separate roots of* $x^2 + bx + c = 0$*, the particular solution would be in form of:*
$$a_{n,p} = g^*(n)r^n$$
with $g^*(n)$ *is a polynomial degree* $k+1$ *of* n*.*
3. *If the constant* r *matches the double root of* $x^2+bx+c=0$*, the particular solution would be in form of:*
$$a_{n,p} = g^*(n)r^n$$
with $g^*(n)$ *is a polynomial degree* $k+2$ *of* n*.*

Proof. Let's look back at our formula (**??**) of $H(n,p)$, and call the product $\prod_{k=1}^{p-1}(n+k)$ to be $C(p)$. We have:

$$\begin{aligned} C(1) &= 1 \\ C(2) &= n+1 \\ C(3) &= (n+1)(n+2) \\ C(4) &= (n+1)(n+2)(n+3) \end{aligned}$$

Note that $C(p)$ is a polynomial degree $p-1$ of n. Next, we can express $g(n)$ in terms of $C(i)$ with $1 \le i \le k+1$. That means:

$$g(n)r^n = \sum_{j=1}^{k+1} c_j H(n,j)$$

with c_j are constants. Then we take the Laplace transform of $g(n)r^n$, we would have the form of:

$$\mathcal{L}\{g(n)r^n\} = \frac{1-e^{-s}}{s} \cdot \sum_{j=1}^{k+1} \frac{c_j}{(1-re^{-s})^j}$$

Note that the constants c_j in this case does need to be the same as earlier equation. Therefore, our equation for particular solution will be:

$$Y_p(s) = \frac{1-e^{-s}}{s} \cdot \sum_{j=1}^{k+1} \frac{c_j e^{-2s}}{(1-re^{-s})^j(1-r_1e^{-s})(1-r_2e^{-s})} \tag{5.4}$$

with r_1, r_2 are 2 roots of the characteristic polynomial: $x^2+bx+c=0$. Note that we do not need to pay attention on how the fractions $\frac{1}{1-r_1e^{-s}}$ and $\frac{1}{1-r_2e^{-s}}$ change because these 2 fractions (or one fraction $\frac{1}{(1-r_1e^{-s})^2}$ in case of double root), will be combined into the homogeneous solution formula. We will use $h(r_1, r_2)$ to describe terms relating r_1, r_2. We just need to pay attention on the power of $1 - re^{-s}$ because these power (the highest power, to be more specific) will determine the degree of the polynomial $g^*(n)$ after we take the Laplace inverse.

1. If r is different from both r_1 and r_2, after using partial fraction for the right side of (**??**) we will have:

$$Y_p(s) = \frac{1-e^{-s}}{s} \cdot \left(h(r_1, r_2) + \sum_{j=1}^{k+1} \frac{c_j^*}{(1-re^{-s})^j} \right)$$

Note that we would have the same form for $Y_p(s)$ if r_1 and r_2 are complex numbers. For more information about partial fraction, see chapter 2.2 of [**?**]. Taking the Laplace inverse will give us (note that the Laplace inverse of $h(r_1, r_2)$ is combined into the homogeneous solution):

$$a_{n,p} = \sum_{j=1}^{k+1} k_j H(n, j)$$

Because $H(n, k+1)$ has the form of a product of r^n and a polynomial degree k of n, we have:

$$a_{n,p} = g^*(n) r^n$$

with $g^*(n)$ is a polynomial degree k of n.

2. If r equals one of the 2 separate roots (r, r_1). Then (**??**) becomes:

$$Y_p(s) = \frac{1-e^{-s}}{s} \cdot \sum_{j=1}^{k+1} \frac{c_j e^{-2s}}{(1-re^{-s})^{j+1}(1-r_1e^{-s})}$$

Note the highest power has increased by 1. After using partial fraction, we have:

$$Y_p(s) = \frac{1-e^{-s}}{s} \cdot \left(\frac{A}{1-r_1e^{-s}} + \sum_{j=1}^{k+2} \frac{c_j^*}{(1-re^{-s})^j} \right)$$

Taking the Laplace inverse will give us:

$$a_{n,p} = \sum_{j=1}^{k+2} k_j H(n,j)$$

Again, use the formula of $H(n,p)$, we will have:

$$a_{n,p} = g^*(n)r^n$$

with $g^*(n)$ is a polynomial degree $k+1$ of n.

3. If r equals the double root, meaning $r = r_1 = r_2$. From (**??**) we have:

$$Y_p(s) = \frac{1-e^{-s}}{s} \cdot \sum_{j=1}^{k+1} \frac{c_j e^{-2s}}{(1-re^{-s})^{j+2}}$$

Using partial fraction will give us:

$$Y_p(s) = \frac{1-e^{-s}}{s} \cdot \sum_{j=1}^{k+3} \frac{c_j^*}{(1-re^{-s})^j}$$

Note that in all 3 cases, the constants c_j^*, k_j are not the same. Now take the Laplace inverse we will have:

$$a_{n,p} = \sum_{j=1}^{k+3} k_j H(n,j)$$

Therefore:

$$a_{n,p} = g^*(n)r^n$$

with $g^*(n)$ is a polynomial degree $k+2$ of n.

□

5.2.3. *Example.* Let's look back to the Example **??**:

$$a_{n+2} - 5a_{n+1} + 6a_n = n2^n$$

Our characteristic polynomial x^2-5x+6 has 2 roots (2,3). Our constant $r = 2$ matches one of the roots. Therefore, our particular solution would be in form of:

$$a_{n,p} = g^*(n)2^n$$

with $g^*(n)$ is a polynomial degree $1+1=2$ of n. Combine together we would have the general solution:

$$a_n = c_1 2^n + c_2 3^n + (c_3 n^2 + c_4 n + c_5) 2^n$$

We can see that this result agrees with the result we had in the example.

5.3. **Forcing function involving sine and cosine.** Our next goal is to deal with forcing functions that involve trigonometric functions sine and cosine.

5.3.1. *Dealing with sine.* Let's look at a general relation:

$$a_{n+2} + b a_{n+1} + c a_n = \sin(kn)$$

First, let's find the Laplace transform function of $\sin(kn)$. We will use the following identity:

$$\sin(ki+k) + \sin(ki-k) = 2\sin(ki)\cos(k)$$

From (**??**), we have:

$$\begin{aligned}
G(s) &= \sum_{i=1}^{\infty} e^{-si} \sin(ki) \\
2\cos(k)G(s) &= \sum_{i=1}^{\infty} e^{-si} \left(2\sin(ki)\cos(k)\right) \\
&= \sum_{i=1}^{\infty} e^{-si} \left(\sin(ki+k) + \sin(ki-k)\right) \\
&= \sum_{i=1}^{\infty} e^{-si} \sin\left(k(i+1)\right) + \sum_{i=1}^{\infty} e^{-si} \sin\left(k(i-1)\right) \\
2\cos(k)G(s) &= \frac{1}{e^{-s}} \cdot \big(G(s) - e^{-s}\sin(k)\big) + e^{-s}G(s) \\
G(s) &= \frac{\sin(k)e^{-s}}{e^{-2s} - 2\cos(k)e^{-s} + 1}
\end{aligned}$$

Therefore:

$$\mathcal{L}\{f(t) = \sin(kn), \quad n \le t < n+1\} = \frac{1-e^{-s}}{s} \cdot \frac{\sin(k)e^{s}}{e^{2s} - 2\cos(k)e^{s} + 1}$$

Now starting from our relation equation, taking the Laplace transform for both sides, we will have:

$$\begin{aligned}
Y(s) &= L(s) + \frac{1-e^{-s}}{s} \cdot \frac{\sin(k)e^{s}}{(e^s-\alpha)(e^s-\beta)(e^s-u)(e^s-v)} \\
Y(s) &= L(s) + Q(s)
\end{aligned}$$

with $(\alpha, \beta) = (\cos(k) + i\sin(k), \cos(k) - i\sin(k))$, which are 2 roots of $x^2 - 2\cos(k)x + 1 = 0$, and $u, \quad v$ are 2 roots of $x^2 + bx + c = 0$ (with $x = e^s$). The function $L(s)$ corresponds to what we call the homogenous solution, and the $Q(s)$ is responsible for the particular solution. Let's calculate the Laplace inverse of $Q(s)$:

$$
\begin{aligned}
Q(s) &= \frac{\sin(k)e^s(1-e^{-s})}{s} \cdot \frac{1}{(e^s-\alpha)(e^s-\beta)(e^s-u)(e^s-v)} \\
&= \frac{\sin(k)e^s(1-e^{-s})}{s} \cdot \\
&\quad \left(\frac{1}{c_1(e^s-\alpha)} + \frac{1}{c_2(e^s-\beta)} + \frac{1}{c_3(e^s-u)} + \frac{1}{c_4(e^s-v)}\right) \\
&= \frac{\sin(k)(1-e^{-s})}{s} \cdot \\
&\quad \left(\frac{1}{c_1(1-\alpha e^{-s})} + \frac{1}{c_2(1-\beta e^{-s})} + \frac{1}{c_3(1-ue^{-s})} + \frac{1}{c_4(1-ve^{-s})}\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}^{-1}\{Q(s)\} &= \sin(k) \cdot \left(\frac{\alpha^n}{c_1} + \frac{\beta^n}{c_2} + \frac{u^n}{c_3} + \frac{v^n}{c_4}\right) \\
\mathcal{L}^{-1}\{Q(s)\} &= o(\alpha^n, \beta^n) + p(u^n, v^n)
\end{aligned}
$$

with:

$$
\begin{aligned}
c_1 &= (\alpha-\beta)(\alpha-u)(\alpha-v) \\
c_2 &= (\beta-\alpha)(\beta-u)(\beta-v) \\
c_3 &= (u-\alpha)(u-\beta)(u-v) \\
c_4 &= (v-\alpha)(v-\beta)(v-u)
\end{aligned}
$$

Let's examine the 2 terms of $o(\alpha^n, \beta^n)$. First of all, De Moivre's formula gives us:

$$\alpha^n = (\cos(k) + i\sin(k))^n = \cos(kn) + i\sin(kn)$$

$$\beta^n = (\cos(k) - i\sin(k))^n = \cos(kn) - i\sin(kn)$$

Next, when we simplify the coefficients of α^n, and β^n (2 denominators), we will see that they have similar forms: $k_1 i + k_2$, and $-k_1 i + k_2$. Using these facts we will have:

$$
\begin{aligned}
o(\alpha^n, \beta^n) &= \sin(k) \cdot \left(\frac{\cos(kn) + i\sin(kn)}{k_1 i + k_2} + \frac{\cos(kn) - i\sin(kn)}{-k_1 i + k_2} \right) \\
&= \frac{\sin(k)}{k_1^2 + k_2^2} \cdot (2k_1 \sin(kn) + 2k_2 \cos(kn)) \\
o(\alpha^n, \beta^n) &= k_3 \sin(kn) + k_4 \cos(kn)
\end{aligned}
$$

Combine this with $p(u^n, v^n)$, which has the form of $k_5 u^n + k_6 v^n$. We now know that our particular solution is in the form of: $k_3 \sin(kn) + k_4 \cos(kn) + k_5 u^n + k_6 v^n$. Finally, combine this particular solution with the homogenous solution, we will have our general solution:

$$a_n = c_1^* u^n + c_2^* v^n + c_3^* \sin(kn) + c_4^* \cos(kn)$$

Note that we have assumed that our characteristic polynomial function has 2 separate complex roots (u, v). The case when it has 1 double root is very similar. Instead of having 2 terms $\frac{1}{1-ue^{-s}}$ and $\frac{1}{1-ve^{-s}}$, we would have 2 terms in form of $\frac{1}{(1-we^{-s})^2}$ and $\frac{1}{1-we^{-s}}$, with w is the double root. Therefore, using the formula for $H(n,p)$ to take the Laplace inverse, our final general solution would be in form of:

$$a_n = (c_1 n + c_2) w^n + c_3 \sin(kn) + c_4 \cos(kn)$$

Example. Let's use our general solution formula to find the closed form for the following sequence:

$$a_{n+2} - 3a_{n+1} + 2a_n = \sin(n); \quad a_0 = 0, \quad a_1 = 1.$$

Our characteristic polynomial $x^2 - 3x + 2$ has two roots $(2, 1)$. Therefore, our general solution would be:

$$a_n = c_1 2^n + c_2 + c_3 \sin(n) + c_4 \cos(n)$$

From our relation equation, we could find a_2 and a_3. Then use these results with our initial conditions a_0 and a_1, we could have a system equation to solve for c_i. In fact, we could find that:

$$
\begin{aligned}
c_1 &= 1 - \frac{\sin(1)}{4\cos(1) - 5} \\
c_2 &= \frac{2\sin(2) - 5\sin(1)}{(\cos(1) - 1) \cdot (8\cos(1) - 10)} - 1 \\
c_3 &= \frac{2\cos(1) - 1}{8\cos(1) - 10} \\
c_4 &= \frac{\sin(2) - 3\sin(1)}{(\cos(1) - 1) \cdot (10 - 8\cos(1))}
\end{aligned}
$$

Using these notation to express our constants would make it so difficult to check our formula. Instead, in order to check it, we could express them as approximate values:

$$a_n = (1.296)2^n - 1.915 - 0.0142\sin(n) + 0.619\cos(n)$$

Now we could easily check our formula, and we can see that this is a correct formula for $\{a_n\}$.

5.3.2. *Dealing with cosine.* The method is exactly the same. There's only a little difference in computing the Laplace transform. We use the following formula for computing $G(s)$:

$$\cos(ki + k) + \cos(ki - k) = 2\cos(ki)\cos(k)$$

Now do exactly what we did to find $G(s)$ for sine, we will have:

$$G(s)\{f(n) = \cos(kn)\} = \frac{1 - \cos(k)e^{-s}}{e^{-2s} - 2\cos(k)e^{-s} + 1}$$

Therefore:

$$\mathcal{L}\{f(t) = \cos(kn), n \leq t < n+1\} = \frac{1 - e^{-s}}{s} \cdot \frac{1 - \cos(k)e^{-s}}{e^{-2s} - 2\cos(k)e^{-s} + 1}$$

Again, use the same method we used for sine, we will have our general solution to be the same as of sine:

$$a_n = c_1 u^n + c_2 v^n + c_3 \sin(kn) + c_4 \cos(kn)$$

Using the results we had for sine and cosine with the Theorem **??** will give us the following theorem:

Theorem 5.6. *The particular solution of a recursion relation of the form:*

$$a_{n+2} + ba_{n+1} + ca_n = f(n)$$

with $f(n)$ is a linear combination of $\sin(kn)$ *and* $\cos(kn)$*, in which k is a real constant, will have the form of:*

$$a_{n,p} = A\sin(kn) + B\cos(kn)$$

with A, B are 2 real constants.

Proof. According to the Theorem, $f(n)$ is a linear combination of $\sin(kn)$ and $\cos(kn)$. This means:

$$f(n) = c_1 \sin(kn) + c_2 \cos(kn)$$

Let $g(n) = c_1 \sin(kn)$, $h(n) = c_2 \cos(kn)$. First, consider the following property: if $a_{n,g}$ is the particular solution for the recursion relation $a_{n+2} + ba_{n+1} + ca_n = f(n)$, then the particular solution of the recursion relation $a_{n+2} + ba_{n+1} + ca_n = pf(n)$, in which p is a real constant, will be $pa_{n,g}$. This property is true by linearity of the recursion relation. Now use this property for $g(n)$ and $h(n)$, we will have their particular solutions, respectively, to be: $c_1(c_3\sin(kn)+c_4\sin(kn))$, $c_2(c_5\sin(kn)+c_6\cos(kn))$. Then use the Theorem **??**, we will have:

$$a_{n,p} = A\sin(kn) + B\cos(kn)$$

with A and B are 2 real constants. $\square$

5.4. **Conclusion.** With all the formulas and theorems we have had so far, we now have enough tool to deal with any linear recursion relations of order 2 with the forcing function of the form:

$$f(n) = \sum g_i(n)r_i^n + \sum p_i\sin(k_in) + \sum q_i\cos(c_in)$$

with $g_i(n)$ are polynomials of n, and r_i, p_i, k_i, q_i, c_i are constants.

Example. We could solve for the following relation using our known formulas and theorems:

$$a_{n+2} + ba_{n+1} + ca_n = (n+2)2^n + (n^2+n+3)3^n + 3\sin(2n) + \cos(5n)$$

with b and c are 2 real constants.

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