Putzer's Method via the $\mathcal{Z}$-Transform

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Outline

Difference Equations

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$$f(n) = f(n-2) + f(n-1), \qquad f(0) = 0, \quad f(1) = 1$$

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The solution is

$$y(k) = a^k y_0$$

Solving for the Solution

$$y(1) = ay(0) \tag{1}$$

$$y(2) = ay(1) \tag{2}$$

$$\vdots$$

$$y(k) = ay(k-1) \tag{3}$$

$$\prod_{i=1}^{k} y(i) = \prod_{i=0}^{k-1} ay(i) \tag{4}$$

$$y(k) \prod_{i=1}^{k-1} y(i) = a^k y(0) \prod_{i=1}^{k-1} y(i) \tag{5}$$

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System of Linear Difference Equations

The population of the American bison is categorized into newborns, yearlings, and adults after k years.

- Every year the newborns are 42% of the adults from the previous year.

$$y_1(k+1) = 0.42y_3(k)$$

- Every year 60% of the newborns become yearlings.

$$y_2(k+1) = 0.60y_1(k)$$

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- Every year 95% of the adults survive.

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$$y_1(k+1) = 0.42y_3(k), \quad y_1(0) = \alpha$$
$$y_2(k+1) = 0.6y_1(k), \quad y_2(0) = \beta$$
$$y_3(k+1) = 0.75y_2(k) + 0.9y_3(k), \quad y_3(0) = \gamma$$

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System of Linear Difference Equations

$$A = \begin{bmatrix} 0 & 0 & 0.42 \\ 0.60 & 0 & 0 \\ 0 & 0.75 & 0.95 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 0.315 & 0.399 \\ 0 & 0 & 0.252 \\ 0.45 & 0.7125 & 0.9025 \end{bmatrix}$$

- Finding powers of A is difficult.
- We need other tools.

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$\mathcal{Z}\{1\}(z)$	$\frac{z}{z-1}$
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$\mathcal{Z}\{y(k+1)\}(z)$	$z\mathcal{Z}\{y(k)\} - y(0)z$

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$$y(k+1) - 5y(k) + 6 = 0, \qquad y(0) = 0$$

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$$\mathcal{Z}^{-1}\{Y(z)\} = \frac{3}{2} - \frac{3}{2}5^k$$

$\mathcal{Z}$-Transform on a System of Difference Equations

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Solve using known methods.

Putzer's Method

Putzer's algorithm solves $\mathbf{y}(k+1) = A\mathbf{y}(k)$.

- A, n xn coefficient matrix
- $\mathbf{y}(k+1)$ and $\mathbf{y}(k)$,n-component column vectors

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$$M_0 = I$$

$$M_i = (A - \lambda_i I)(A - \lambda_{i-1} I) \ldots (A - \lambda_1 I)$$

Putzer's Algorithm

$$c_1(k+1) = \lambda_1 c_1(k), \quad c_1(0) = 1$$

$$c_2(k+1) = \lambda_2 c_2(k) + c_1(k), \quad c_2(0) = 0$$

$$c_3(k+1) = \lambda_3 c_3(k) + c_2(k), \quad c_3(0) = 0$$

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$$c_n(k+1) = \lambda_n(k) + c_{n-1}(k), \quad c_n(0) = 0$$

Putzer's Algorithm Example

$$\mathbf{y}(k+1) = A\mathbf{y}(k), \qquad A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad \mathbf{y}(0) = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$p_A(\lambda) = |A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{vmatrix} = (\lambda+1)(\lambda+2) = 0$$

$$\lambda_1 = -1, \quad \lambda_2 = -2$$

$$M_0 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad M_1 = [A - \lambda_1 I] = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}$$

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$$c_1(k+1) = \lambda_1 c_1(k), \quad c_1(0) = 1 \tag{7}$$

$$c_1(k+1) = -c_1(k) \tag{8}$$

$$\mathcal{Z}\{c_1(k+1)\}(z) = -\mathcal{Z}\{c_1(k)\}(z) \tag{9}$$

$$C_1(z) - zc_1(0) = -C_1(z) \tag{10}$$

$$C_1(z) = \frac{z}{z+1} \tag{11}$$

$$\mathcal{Z}^{-1}\{C_1(z)\} = (-1)^k \tag{12}$$

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Final solution:

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The rest of the solutions can be checked inductively.

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Need new closed form solution.

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Defining the New Coefficients

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$$\vdots$$

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Example

$$\mathbf{y}(k+1) = A\mathbf{y}(k), \qquad A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad \mathbf{y}(0) = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$p_A(\lambda) = |A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{vmatrix} = (\lambda+1)(\lambda+2) = 0$$

$$\lambda_1 = -1, \quad \lambda_2 = -2$$

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$$\mathbf{y}(k+1) = A\mathbf{y}(k), \qquad A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad \mathbf{y}(0) = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

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Final solution:

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Thank You!

Resolvent Matrix

$$y_1(k+1) = 0.42y_3(k)$$

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Resolvent Matrix

$$z(zI - A)^{-1} = \sum_{j=0}^{n-1} R_{j+1}(z) P_j$$

Resolvent Matrix

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$$\begin{aligned}\frac{1}{z}(zI-A)R_{j+1}(s)P_j &= \frac{1}{z}(((zI-\lambda_{j+1}I)-(A-\lambda j+1I))R_{j+1}(s)R_{j+1}\\ &= \frac{1}{z}((z-\lambda_{j+1})R_{j+1}(s)P_j-R_{j+1}(s)(A-\lambda_j+1\\ &= \frac{1}{z}(R_{j+1}(s)P_j-R_{j+1}(s)P_{j+1})\end{aligned}$$

$$z(zI-A)Y(s) = \sum_{j=0}^{n-1}\frac{(zI-A)R_{j+1}(z)P_j}{z}$$

$$\sum^{n-1} R_j(z)P_j - R_{j+1}(z)P_{j+1}$$