Device Constructions with Hyperbolas

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Hyperbola Definition

- Conic Section
Hyperbola Definition

- Conic Section
- Two Foci
- Focus and Directrix
The Project

- Basic constructions
- Constructing a Hyperbola
- Advanced constructions
Theorem

Given a circle centered at a point $A$ with radius $r$ and any point $C$ different from $A$, it is possible to construct a circle centered at $C$ that is congruent to the circle centered at $A$ with a compass and straightedge.
Angle Duplication

Device Constructions with Hyperbolas
Angle Duplication

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Constructing a Perpendicular

Device Constructions with Hyperbolas
Device Constructions with Hyperbolas
We Need to Draw a Hyperbola!

- Trisection of an angle and doubling the cube cannot be accomplished with a straightedge and compass.
We Need to Draw a Hyperbola!

- Trisection of an angle and doubling the cube cannot be accomplished with a straightedge and compass.
- We needed a way to draw a hyperbola.
We Need to Draw a Hyperbola!

- Trisection of an angle and doubling the cube cannot be accomplished with a straightedge and compass.
- We needed a way to draw a hyperbola.
- Items we needed:
  - one cork board
  - one poster board
  - one pair of scissors
  - one roll of string
  - a box of push pins
  - some paper if you do not already have some
  - a writing utensil
  - some straws, which we picked up at McDonald’s
The Device for Drawing Hyperbolas
$R = \text{length of tube}$

$C = \text{length of string}$
\[ C = PF_1 + (R - PF_2) + R \]
\[ PF_1 - PF_2 = C - 2R \]
Lemma

Let \( \triangle ABP \) be a triangle with the following property: point \( P \) lies along the hyperbola with eccentricity 2, \( B \) as its focus, and the perpendicular bisector of \( AB \) as its directrix. Then \( \angle B = 2 \angle A \).
Proof.

\[ h^2 = a^2 - (c - b)^2 \]
\[ h^2 = g^2 - (c + b)^2 \]
\[
\begin{align*}
\frac{a}{b} &= 2 \\
\frac{2 \left( \frac{h}{g} \right) \left( \frac{b + c}{g} \right)}{2 \sin(\angle A) \cos(\angle A)} &= \sin(\angle B) \\
\sin(2\angle A) &= \sin(\angle B) \\
2\angle A &= \angle B
\end{align*}
\]
Lemma

Let \( \triangle ABP \) be a triangle such that \( \angle B = 2\angle A \). Then point \( P \) lies along the hyperbola with eccentricity 2, \( B \) as its focus, and the perpendicular bisector of \( AB \) as its directrix.
Proof.

\[ 2 \angle A = \angle B \]
\[ \sin(2\angle A) = \sin(\angle B) \]
\[ 2 \sin(\angle A) \cos(\angle A) = \sin(\angle B) \]
\[ 2 \left( \frac{h}{g} \right) \left( \frac{b + c}{g} \right) = \frac{h}{a} \]
\[ \ldots \]
\[ (a - 2b)(2c - a) = 0 \]
\[ \frac{a}{b} = 2 \]
Theorem

Let $\overline{AB}$ be a fixed line segment. Then the locus of points $P$ such that $\angle PBA = 2\angle PAB$ is a hyperbola with eccentricity $2$, with focus $B$, and the perpendicular bisector of $\overline{AB}$ as its directrix.
Let \( O \) denote the vertex of the angle.

Use a compass to draw a circle centered at \( O \), and obtain the points \( A \) and \( B \) on the angle.

Construct the hyperbola with eccentricity \( \epsilon = 2 \), focus \( B \), and directrix the perpendicular bisector of \( AB \).

Let this hyperbola intersect the circle at \( P \).

Then \( OP \) trisects the angle.
Trisecting the Angle

- Given an angle $\angle O$, mark a point $A$ on the given rays.
Trisecting the Angle

- Draw a circle, centered at $O$ with radius $OA$. Mark the intersection on the second ray $B$, and draw the segment $AB$. 

![Diagram of trisecting an angle]
Trisecting the Angle

- Draw a circle, centered at $O$ with radius $OA$. Mark the intersection on the second ray $B$, and draw the segment $AB$. 
Divide the segment $\overline{AB}$ into 6 equal parts: to do this, we pick a point $G_1$, not on $\overline{AB}$, and draw the ray $\overline{AG_1}$. Mark points $G_2$, $G_3$, $G_4$, $G_5$, and $G_6$ on the ray such that:

$$\overline{AG_1} = \overline{G_1G_2} = \overline{G_2G_3} = \overline{G_3G_4} = \overline{G_4G_5} = \overline{G_5G_6}.$$
Draw $G_6B$. Draw lines through $G_1$, $G_2$, $G_3$, $G_4$ and $G_5$ parallel to $G_6B$. Each intersection produces equal length line segments on $AB$. Mark each intersection as shown, and treat each segment as a unit length of one.
Extend $\overline{AB}$ past $A$ a length of 2 units as shown below. Mark this point $F_2$. Construct a line perpendicular to $AB$ through the point $D_1$. Using $F_2$ and $B$ as the foci and $V$ as the vertex, use the device to construct a hyperbola, called $h$. Since the distance from the center, $C$, to $F_1$ is 4 units and the distance $C$ to the vertex, $V$, is 2 units, the hyperbola has eccentricity of 2 as required.
Mark the intersection point between the hyperbola, \( h \), and the circle \( \hat{OA} \) as \( P \). Draw the segment \( \overline{OP} \). The angle \( \angle POB \) trisects \( \angle AOB \).
Constructing $\sqrt[3]{2}$

Start with a given unit length of $\overline{AB}$.

$\overline{AB}$
Constructing $\sqrt[3]{2}$

Start with a given unit length of $\overline{AB}$.

Construct a square with side $\overline{AB}$ and mark the point shown.
Draw a line $l$ through the points $A$ and $E$. Extend line $\overline{AB}$ past $B$ a unit length of $\overline{AB}$. Draw a circle, centered at $A$, with radius $\overline{AC}$ and mark the intersection on $l$ as $V$. Draw a circle centered at $E$ with radius $\overline{AE}$ and mark the intersection on $l$ as $F_1$. Draw the circle centered at $A$ with radius $\overline{AF_1}$ and mark that intersection on $l$ as $F_2$. Bisect the segment $\overline{EB}$ and mark the point $O$. 

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Draw a circle centered at $O$ with a radius of $\overline{OA}$. Using the device, draw a hyperbola with foci $F_1$ and $F_2$ and vertex $V$. 
The circle intersects the hyperbola twice. Mark the leftmost intersection $X$ and draw a perpendicular line from $\overline{AC}$ to $X$. This segment has length $\sqrt{2}$. 

![Diagram of circle intersecting hyperbola]
Construction Proof

- We can easily prove that the above construction is valid if we translate the above into Cartesian coordinates.
- If we allow the point $A$ to be treated as the origin of the $x$-$y$ plane and $B$ be the point $(1, 0)$, we can write the equations of the circle and hyperbola.
- The circle is centered at 1 unit to the right and $\frac{1}{2}$ units up, giving it a center of $(1, \frac{1}{2})$ and a radius of $\sqrt{\frac{5}{4}}$. This gives the circle the equation:

$$ (x - 1)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{5}{4}. $$
The hyperbola, being rectangular with vertex \((\sqrt{2}, \sqrt{2})\), has the equation \(xy = 2\), so \(y = \frac{2}{x}\).

Substituting this expression into the circle’s equation and solving for \(x\) yields the following:

\[
(x - 1)^2 + \left(\frac{2}{x} - \frac{1}{2}\right)^2 = \frac{5}{4}
\]
\[
x^2 - 2x + \frac{4}{x^2} - \frac{2}{x} = 0
\]
\[
x^4 - 2x - 2x^3 + 4 = 0
\]
\[
(x^3 - 2)(x - 2) = 0
\]

From here we can see that both \(x = \sqrt[3]{2}\) and \(x = 2\) are solutions.

This proves that the horizontal distance from the \(y\)-axis to the point \(X\) is \(\sqrt[3]{2}\).
Bibliography

- [Compass equivalence theorem](http://en.wikipedia.org/wiki/Compass_equivalence_theorem)
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