Device Constructions with Hyperbolas

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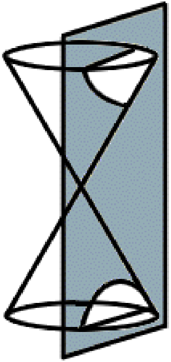
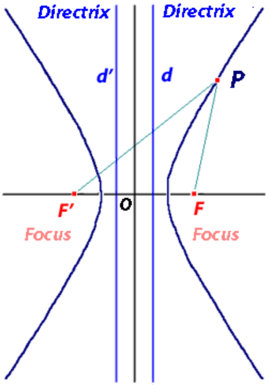
Hyperbola Definition

- Conic Section



Hyperbola Definition

- Conic Section
- Two Foci
- Focus and Directrix



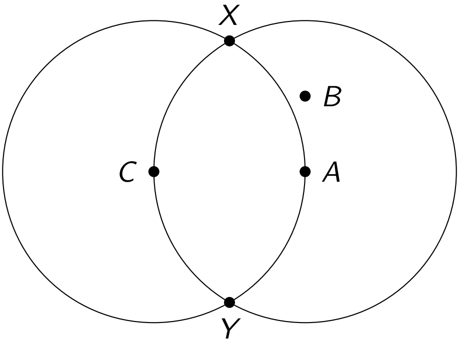
The Project

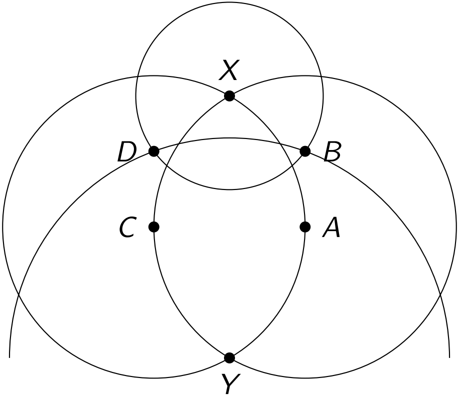
- Basic constructions
- Constructing a Hyperbola
- Advanced constructions

Rusty Compass

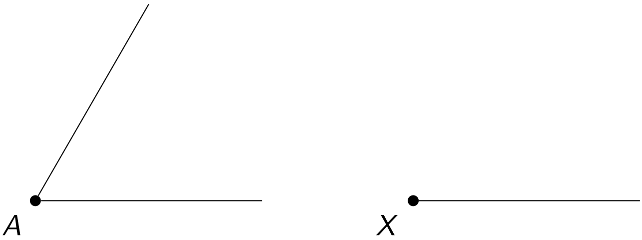
Theorem

Given a circle centered at a point A with radius r and any point C different from A, it is possible to construct a circle centered at C that is congruent to the circle centered at A with a compass and straightedge.

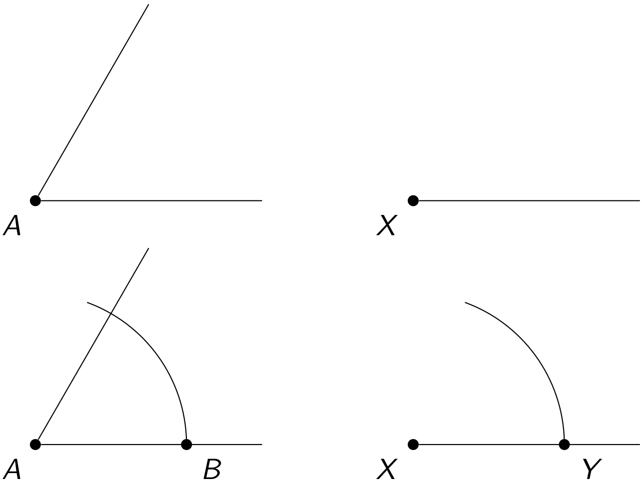

X
B
C
A
Y

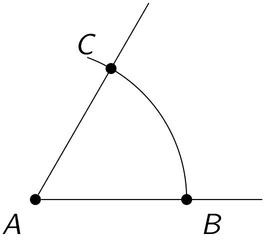
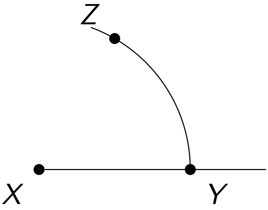

X
D
B
C
A
Y

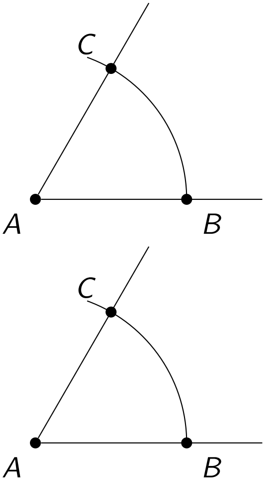
Angle Duplication



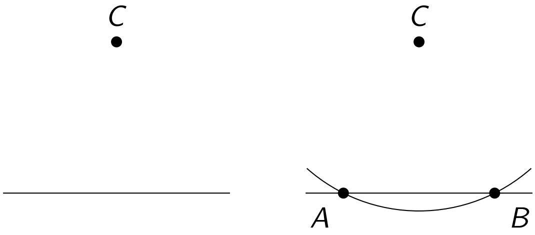
Angle Duplication

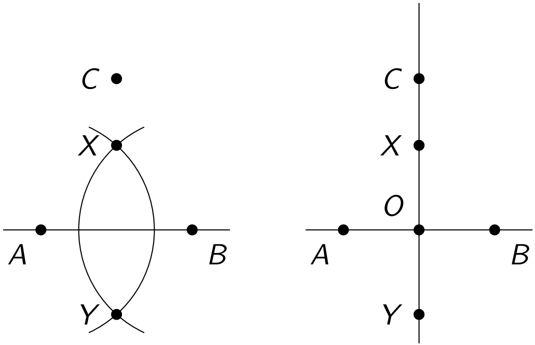



C
A
B

Z
X
Y


C
A
B
Z
X
Y
C
A
B
Z
X
Y

Constructing a Perpendicular




C
X
A
B
Y
C
X
O
A
B
Y

We Need to Draw a Hyperbola!

- Trisection of an angle and doubling the cube cannot be accomplished with a straightedge and compass.

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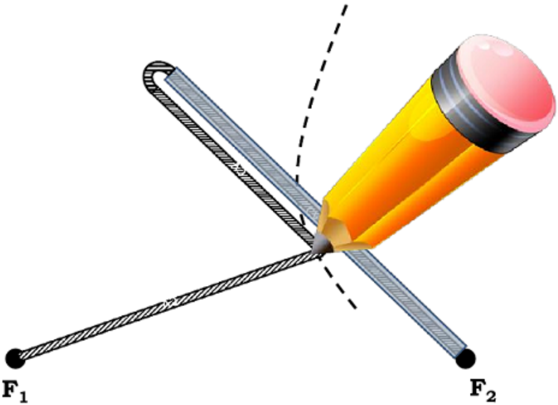
- Trisection of an angle and doubling the cube cannot be accomplished with a straightedge and compass.
- We needed a way to draw a hyperbola.

We Need to Draw a Hyperbola!

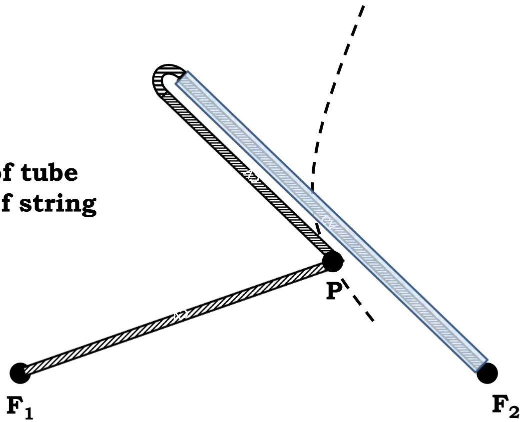
- Trisection of an angle and doubling the cube cannot be accomplished with a straightedge and compass.
- We needed a way to draw a hyperbola.
- Items we needed:
  - one cork board
  - one poster board
  - one pair of scissors
  - one roll of string
  - a box of push pins
  - some paper if you do not already have some
  - a writing utensil
  - some straws, which we picked up at McDonald's

The Device

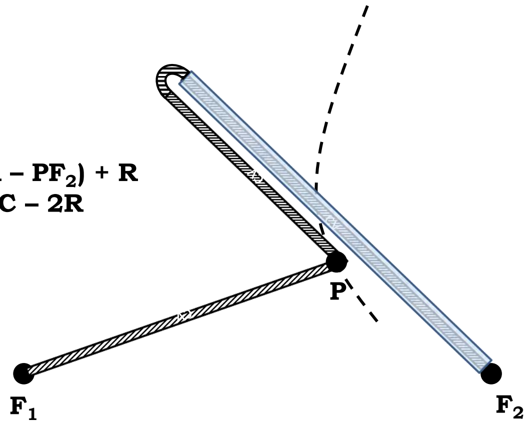


The Device for Drawing Hyperbolas

R = length of tube
C = length of string



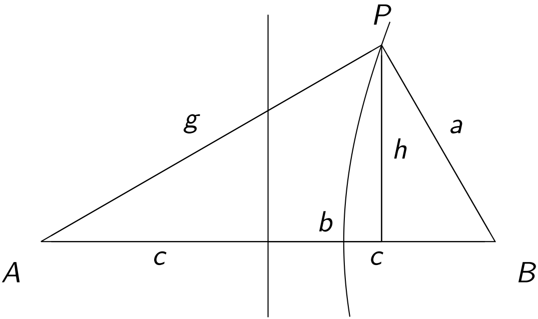
$$C = PF_1 + (R - PF_2) + R$$
$$PF_1 - PF_2 = C - 2R$$



Hyperbolas and Triangles

Lemma

Let $\triangle ABP$ be a triangle with the following property: point P lies along the hyperbola with eccentricity 2, B as its focus, and the perpendicular bisector of $\overline{AB}$ as its directrix. Then $\angle B = 2\angle A$.

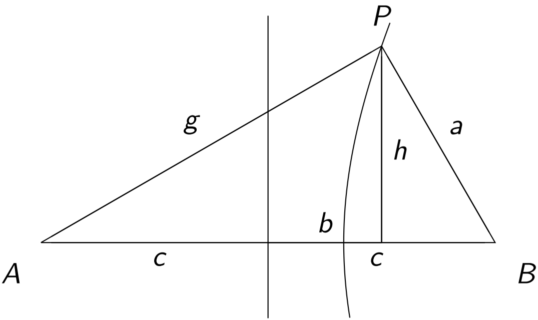


Proof.

$$
\begin{aligned}
h^2 &= a^2 - (c-b)^2 \\
h^2 &= g^2 - (c+b)^2 \\
&\cdots \\
\frac{a}{b} &= 2 \\
&\cdots \\
2\left(\frac{h}{g}\right)\left(\frac{b+c}{g}\right) &= \frac{h}{a} \\
2\sin(\angle A)\cos(\angle A) &= \sin(\angle B) \\
\sin(2\angle A) &= \sin(\angle B) \\
2\angle A &= \angle B
\end{aligned}
$$

Lemma

Let $\triangle ABP$ be a triangle such that $\angle B = 2\angle A$. Then point P lies along the hyperbola with eccentricity 2, B as its focus, and the perpendicular bisector of $\overline{AB}$ as its directrix.



Proof.

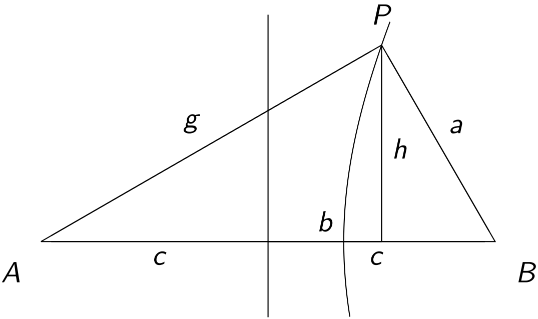
$$
\begin{aligned}
2\angle A &= \angle B \\
\sin(2\angle A) &= \sin(\angle B) \\
2\sin(\angle A)\cos(\angle A) &= \sin(\angle B) \\
2\left(\frac{h}{g}\right)\left(\frac{b+c}{g}\right) &= \frac{h}{a} \\
&\cdots \\
(a-2b)(2c-a) &= 0 \\
\frac{a}{b} &= 2
\end{aligned}
$$

$\square$

Result

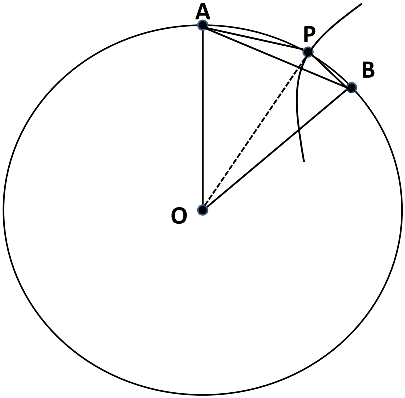
Theorem

Let $\overline{AB}$ be a fixed line segment. Then the locus of points P such that $\angle PBA = 2\angle PAB$ is a hyperbola with eccentricity 2, with focus B, and the perpendicular bisector of $\overline{AB}$ as its directrix.



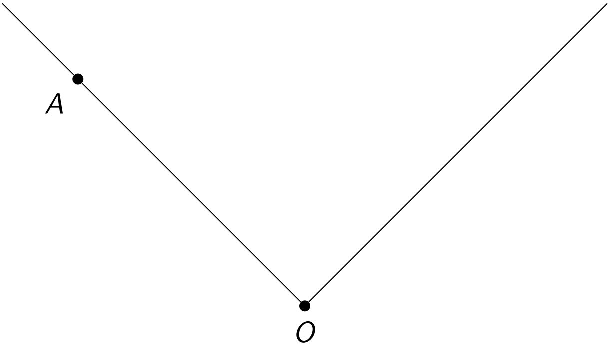
Trisecting the Angle - The Classical Construction

- Let O denote the vertex of the angle.
- Use a compass to draw a circle centered at O, and obtain the points A and B on the angle.
- Construct the hyperbola with eccentricity $\epsilon = 2$, focus B, and directrix the perpendicular bisector of $\overline{AB}$.
- Let this hyperbola intersect the circle at P.
- Then OP trisects the angle.



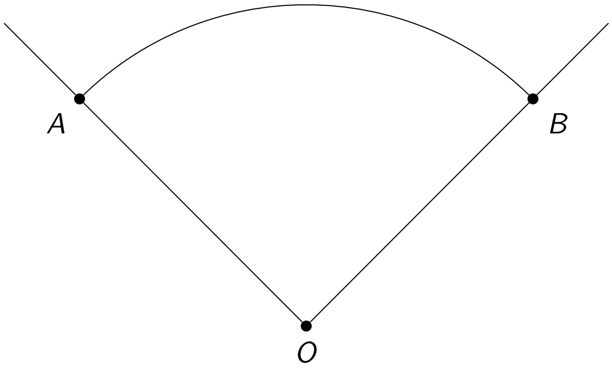
Trisecting the Angle

- Given an angle $\angle O$, mark a point A on on the the given rays.



Trisecting the Angle

- Draw a circle, centered at O with radius $\overline{OA}$. Mark the intersection on the second ray B, and draw the segment $\overline{AB}$.



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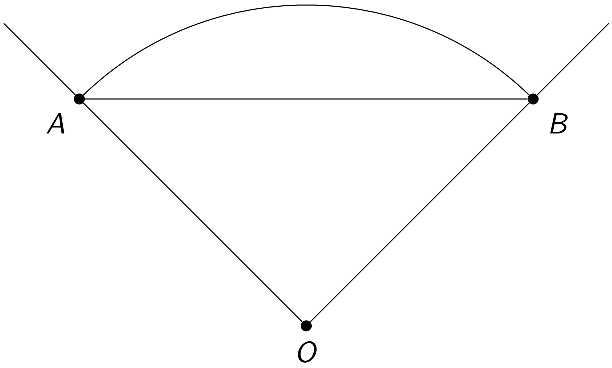
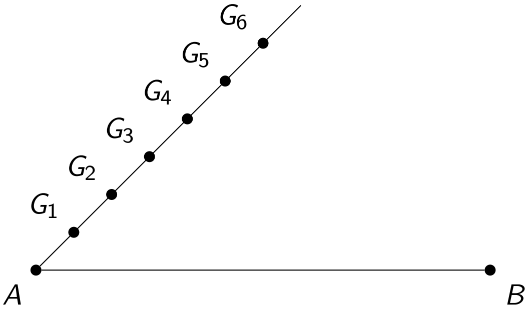
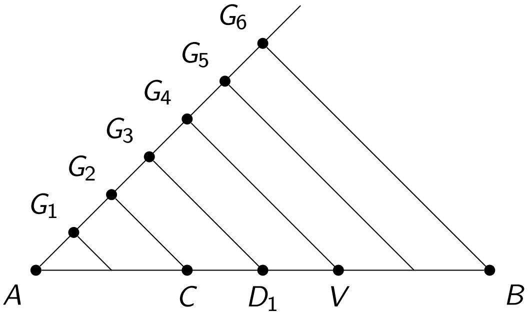


Divide the segment $\overline{AB}$ into 6 equal parts: to do this, we pick a point G_1, not on $\overline{AB}$, and draw the ray $\overline{AG_1}$. Mark points G_2, G_3, G_4, G_5, and G_6 on the ray such that:

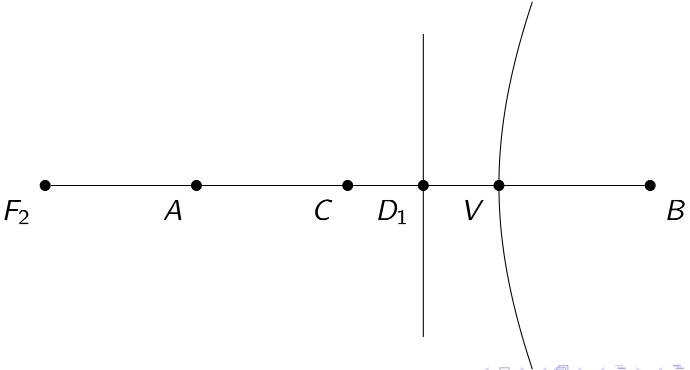
$$\overline{AG_1} = \overline{G_1 G_2} = \overline{G_2 G_3} = \overline{G_3 G_4} = \overline{G_4 G_5} = \overline{G_5 G_6}.$$



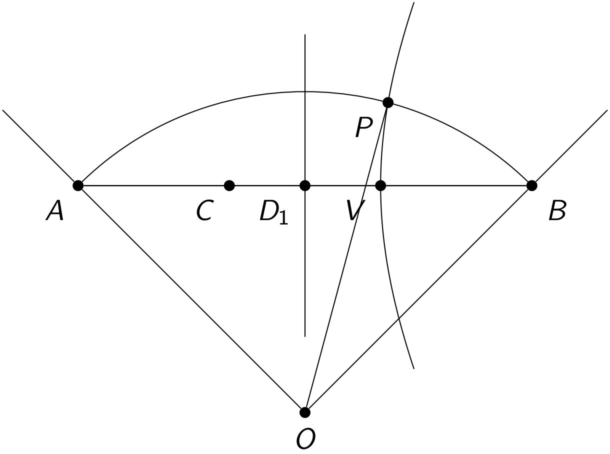
Draw $\overline{G_6 B}$. Draw lines through G_1, G_2, G_3, G_4 and G_5 parallel to $\overline{G_6 B}$. Each intersection produces equal length line segments on $\overline{AB}$. Mark each intersection as shown, and treat each segment as a unit length of one.



Extend $\overline{AB}$ past A a length of 2 units as shown below. Mark this point F_2. Construct a line perpendicular to AB through the point D_1. Using F_2 and B as the foci and V as the vertex, use the device to construct a hyperbola, called h. Since the distance from the the center, C, to F_1 is 4 units and the distance C to the vertex, V, is 2 units, the hyperbola has eccentricity of 2 as required.

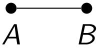


Mark the intersection point between the hyperbola, h, and the circle $\widehat{OA}$ as P. Draw the segment $\overline{OP}$. The angle $\angle POB$ trisects $\angle AOB$.



Constructing $\sqrt[3]{2}$

Start with a given unit length of $\overline{AB}$.



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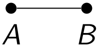
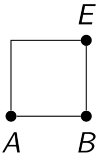
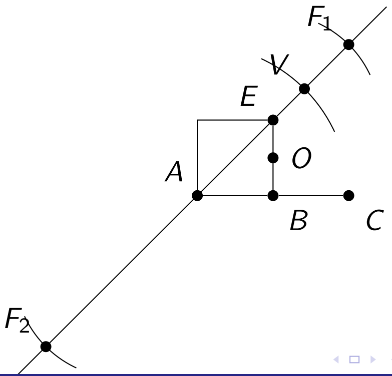


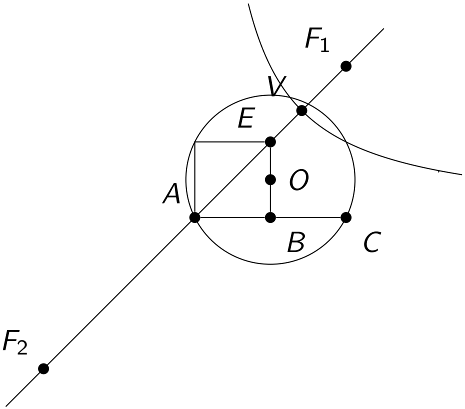
Construct a square with side $\overline{AB}$ and mark the point shown.



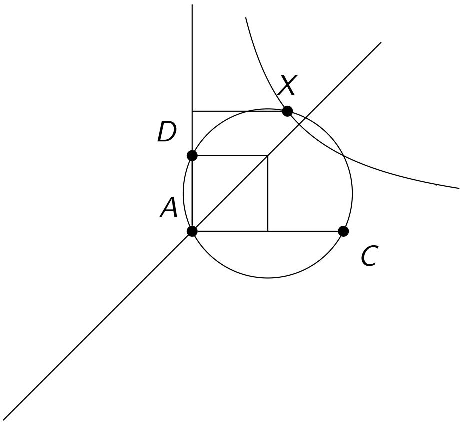
Draw a line l through the points A and E. Extend line $\overline{AB}$ past B a unit length of $\overline{AB}$. Draw a circle, centered at A, with radius $\overline{AC}$ and mark the intersection on l as V. Draw a circle centered at E with radius $\overline{AE}$ and mark the intersection on l as F_1. Draw the circle centered at A with radius $\overline{AF_1}$ and mark that intersection on l as F_2. Bisect the segment $\overline{EB}$ and mark the point O.



Draw a circle centered at O with a radius of $\overline{OA}$. Using the device, draw a hyperbola with foci F_1 and F_2 and vertex V.



The circle intersects the hyperbola twice. Mark the leftmost intersection X and draw a perpendicular line from $\overline{AC}$ to X. This segment has length $\sqrt[3]{2}$.



Construction Proof

- We can easily prove that the above construction is valid if we translate the above into Cartesian coordinates.
- If we allow the point A to be treated as the origin of the x - y plane and B be the point $(1, 0)$, we can write the equations of the circle and hyperbola.
- The circle is centered at 1 unit to the right and $\frac{1}{2}$ units up, giving it a center of $(1, \frac{1}{2})$ and a radius of $\sqrt{\frac{5}{4}}$. This gives the circle the equation:

$$(x-1)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{5}{4}.$$

- The hyperbola, being rectangular with vertex $(\sqrt{2}, \sqrt{2})$, has the equation $xy = 2$, so $y = 2/x$.
- Substituting this expression into the circle's equation and solving for x yields the following:

$$\begin{aligned}
(x-1)^2 + \left(\frac{2}{x} - \frac{1}{2}\right)^2 &= \frac{5}{4} \\
x^2 - 2x + \frac{4}{x^2} - \frac{2}{x} &= 0 \\
x^4 - 2x - 2x^3 + 4 &= 0 \\
(x^3 - 2)(x - 2) &= 0
\end{aligned}$$

- From here we can see that both $x = \sqrt[3]{2}$ and $x = 2$ are solutions.
- This proves that the horizontal distance from the y-axis to the point X is $\sqrt[3]{2}$.

Bibliography

- Heath, T., *A History of Greek Mathematics*, Dover Publications, New York, 1981.
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- http://en.wikipedia.org/wiki/Compass_equivalence_theorem

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TIGER TRAILS