DEVICE CONSTRUCTIONS WITH HYPERBOLAS

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Abstract. In this paper we summarize a few geometric constructions using hyperbolas. In addition to a straightedge and compass, we have a device that can draw a hyperbola given the two foci. We describe the device in detail and then use it to perform two constructions, including a classical trisection construction by Pappus.

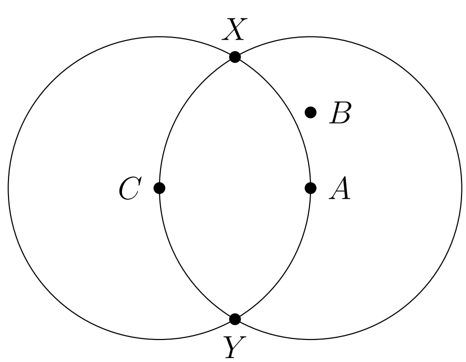
1. Introduction

In order to proceed, we will require a theorem and three basic constructions.

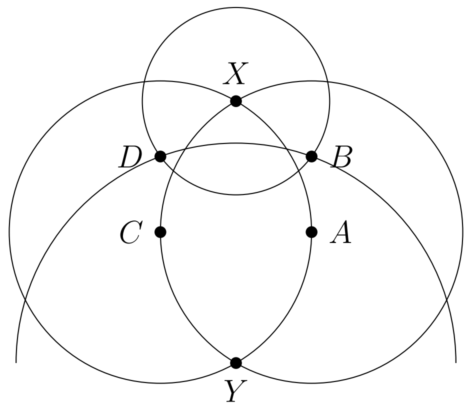
1.1. **Rusty Compass Theorem.**

Theorem 1.1. *Given a circle centered at a point A with radius r and any point C different from A, it is possible to construct a circle centered at C that is congruent to the circle centered at A with a compass and straightedge.*

Proof. The following proof [3] uses only a collapsing compass to duplicate a line segment at another point. Given segment $\overline{AB}$ to copy at point C, construct a circle at A which passes through C and vice versa. Label the intersection points X and Y.



Now draw circles centered at X and Y which both run through point B. Label the other intersection point D. Since C is the reflection of A across $\overline{XY}$ and D is the reflection of B, $\overline{CD} = \overline{AB}$.

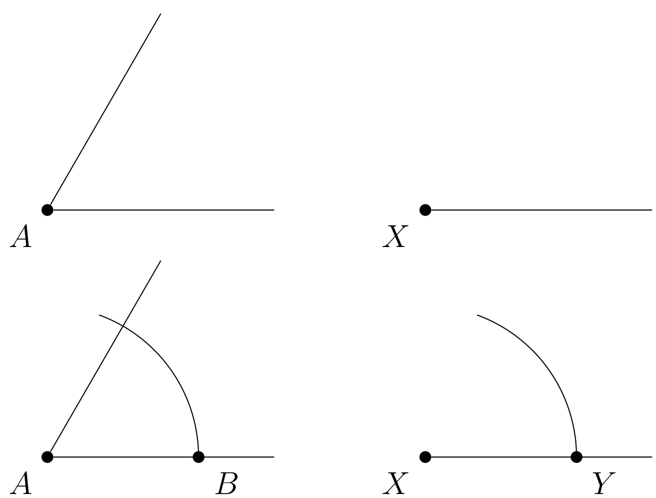


□

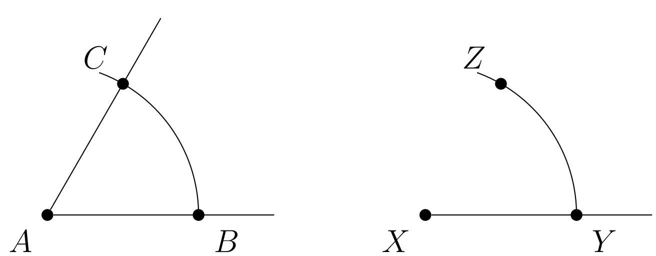
1.2. **Copying an Angle.**

Theorem 1.2. *Given an angle, it is possible to construct a congruent angle with a compass and straightedge.*

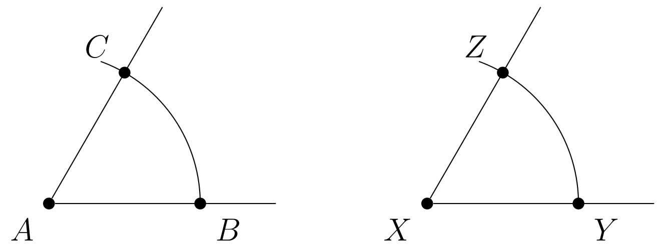
Proof. An angle can be copied as follows:



$$\overline{AB} = \overline{XY}$$



$$\overline{BC} = \overline{YZ}$$



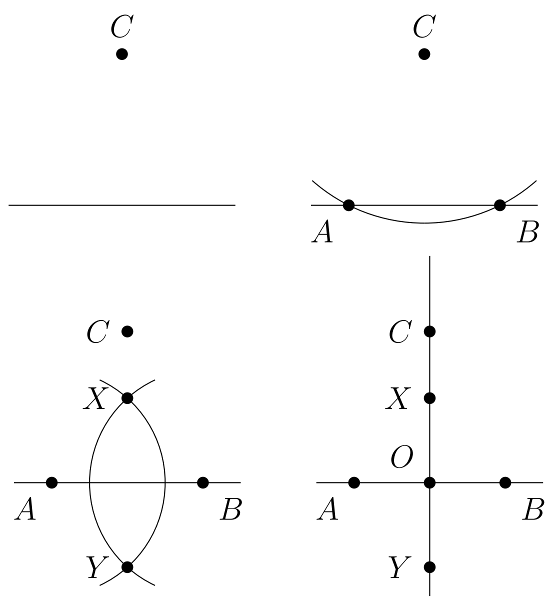
$$\overline{AC} = \overline{XZ}$$

It follows that $\triangle ABC = \triangle XYZ$, and so $\angle A = \angle X$. □

1.3. **Constructing a Perpendicular.**

Theorem 1.3. *Given a line and a point not on the line, it is possible to construct a perpendicular line which runs through the points.*

Proof. The perpendicular line can be constructed in the following manner:



$\overline{AX} = \overline{AB} = \overline{AY} = \overline{YB}$, so quadrilateral $AXBY$ is a rhombus. It follows that $\overline{AB} \perp \overline{XY}$. The line $\overline{XY}$ runs through C since the points X, Y, and Z are all equidistant from A and B.

□

2. Trisecting an Angle

2.1. **The Classical Construction.** In what follows, we discuss the classical construction by Pappus, which uses a hyperbola to trisect an angle.

Lemma 2.1. *Let $\triangle ABP$ be a triangle with the following property: the point P lies on the hyperbola with eccentricity 2, B as its focus, and the perpendicular bisector of $\overline{AB}$ as its directrix. Then:*

$$\angle PBA = 2\angle PAB.$$

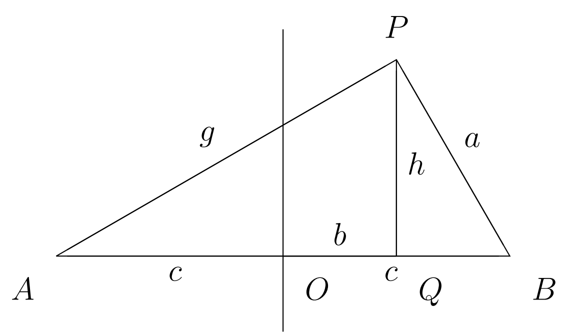


Figure 1

Proof. Let O be the point where the perpendicular bisector of $\overline{AB}$ intersects $\overline{AB}$. We drop a perpendicular from P onto $\overline{AB}$, and obtain the point Q. We label the lengths of certain segments as in Figure 2.1.

By the Pythagorean Theorem in the triangles $\triangle PBQ$ and $\triangle APQ$:

$$\begin{aligned} h^2 &= a^2 - (c-b)^2 \\ h^2 &= g^2 - (c+b)^2 \end{aligned}$$

Thus we have the equality:

$$\begin{aligned} a^2 - (c-b)^2 &= g^2 - (c+b)^2 \\ a^2 - (c^2 - 2bc + b^2) &= g^2 - (c^2 + 2bc + b^2) \\ a^2 + 4bc &= g^2 \end{aligned}$$

If $\angle B$ is actually obtuse, then the first line would read instead:

$$h^2 = a^2 - (b-c)^2,$$

and the result would be the same.

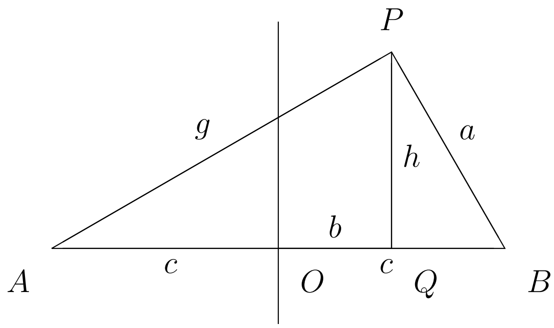


FIGURE 2

It is given that the eccentricity of the hyperbola is 2. So, with the knowledge that $\frac{a}{b} = 2$, we find that:

$$\begin{aligned}
2ab + 4bc &= g^2 \\
2a\left(b + 2\frac{bc}{a}\right) &= g^2 \\
2a(b+c) &= g^2 \\
\frac{2(b+c)}{g^2} &= \frac{1}{a} \\
\frac{2h(b+c)}{g^2} &= \frac{h}{a} \\
2\left(\frac{h}{g}\right)\left(\frac{b+c}{g}\right) &= \frac{h}{a} \\
2\sin(\angle A)\cos(\angle A) &= \sin(\angle B) \\
\sin(2\angle A) &= \sin(\angle B) \\
2\angle A &= \angle B
\end{aligned}$$

□

In fact, the converse also holds:

Lemma 2.2. *Let the triangle $\triangle ABP$ be such that $\angle PBA = 2\angle PAB$. Then the point P lies on the hyperbola with eccentricity 2, B as its focus, and the perpendicular bisector of $\overline{AB}$ as its directrix.*

Proof. We refer to the same Figure 2.1, which we repeat here for convenience.

By hypothesis, $2\angle A = \angle B$. We then have:

$$
\begin{aligned}
\sin(2\angle A) &= \sin(\angle B) \\
2\sin(\angle A)\cos(\angle A) &= \sin(\angle B) \\
2\left(\frac{h}{g}\right)\left(\frac{b+c}{g}\right) &= \frac{h}{a} \\
2a(b+c) &= g^2 \\
&= (b+c)^2 + h^2 \\
&= (b+c)^2 + a^2 - (c-b)^2 \\
2a(b+c) &= 4bc + a^2
\end{aligned}
$$

If $\angle B$ is actually obtuse, then the penultimate line would read instead:

$$(b+c)^2 + a^2 - (b-c)^2,$$

and the result would be the same.

Continuing from the last line:

$$
\begin{aligned}
2ac - a^2 - 4bc + 2ab &= 0 \\
(a-2b)(2c-a) &= 0
\end{aligned}
$$

So, at least one of these factors must be 0. If $2c - a = 0$, then $\triangle ABP$ must be right isosceles and $b = c$. Thus, $a = 2b$ and $\frac{a}{b} = 2$. On the other hand, if $a - 2b = 0$, then it immediately follows that $\frac{a}{b} = 2$. In all cases, the lengths a and b form a constant ratio of 2, thus the point P lies on the hyperbola of eccentricity 2 with point B as its focus and the perpendicular bisector of $\overline{AB}$ as its directrix. □

With these two lemmas, we really proved the following theorem:

Theorem 2.3. *Let $\overline{AB}$ be a fixed line segment. Then the locus of points P such that $\angle PBA = 2\angle PAB$ is a hyperbola with eccentricity 2, with focus B, and the perpendicular bisector of $\overline{AB}$ as its directrix.*

Proof. □

2.2. **Trisecting the Angle.** Using the what we now know from Theorem 2.3, we can show that a trisection is possible.

Suppose we have an angle, and we label its vertex O. The following construction trisects this angle:

(1) Using a compass, draw a circle centered at O and obtain the points A and B on the sides of the angle.
(2) We draw the line segment $\overline{AB}$.
(3) We draw the hyperbola with focus B, eccentricity 2 and the perpendicular bisector of $\overline{AB}$ as its directrix.

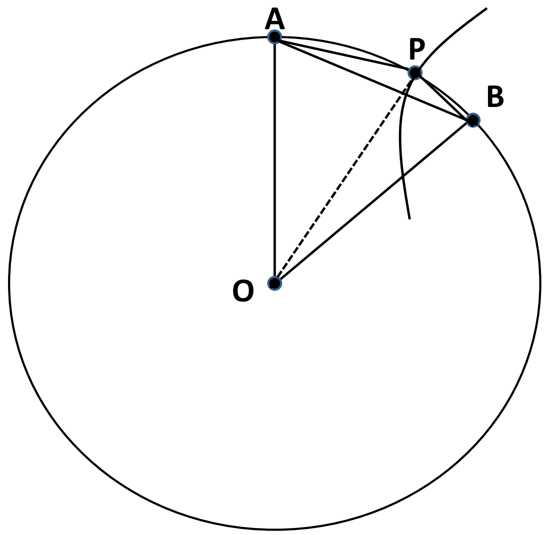


FIGURE 3. Trisecting the Angle using a Hyperbola

(4) Let this hyperbola intersect the circle in the point P.
(5) The segment $\overline{OP}$ trisects the angle $\angle AOB$.

Since $\angle PAB$ is on the circle and shares the same arc with the central angle $\angle POB$, $\angle POB$ is twice that of $\angle PAB$. Similarly, $\angle PBA$ is twice that of $\angle POA$. Therefore, since $\angle PBA = 2\angle PAB$ (by Theorem 2.3), that makes $\angle POA = 2\angle POB$. Thus, $\angle AOB = 3\angle POB$ and $\overline{OP}$ trisects the angle.

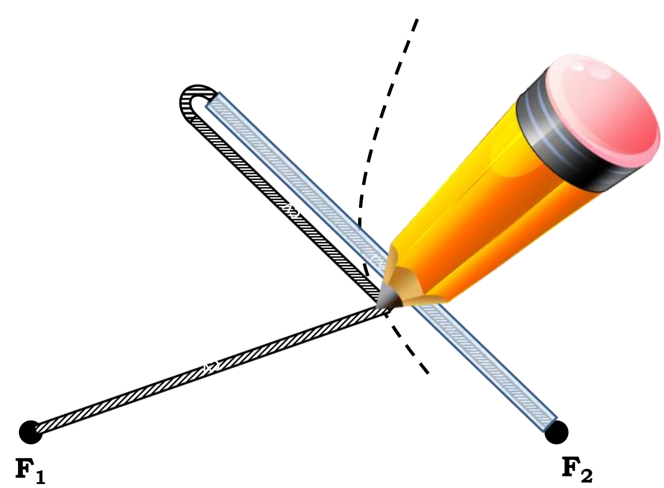
Of course, the difficult part is now constructing the given hyperbola. For this, we use the device described in the following section.

2.3. **The Quest to Draw a Hyperbola.** The authors were sitting around one day talking about math and, all of a sudden, Alfonso shouted out that we needed to draw a hyperbola. The problem is that all the power was out so we couldn't use any of the computers. So we decided to go and buy a device that draws hyperbolas, and where do you go at three in the morning to buy a hyperbola maker? Why Walmart of course. But to our misfortune, when we arrived at Walmart, they were completely sold out of hyperbola makers. There wasn't a display model we could get or even an old one in the back storage. All hope was lost, until we remembered that prior to the power going out we were watching a McGyver marathon and decided that if they were sold out that we could just make one.

The list of things we could scrounge up around Walmart were:

- one cork board
- one poster board
- one pair of scissors
- one roll of string
- a box of push pins
- some paper if you do not already have some
- a writing utensil
- some straws, which we picked up at McDonald's

First we cut the poster board to the size of the cork board and pinned it down with the push pins. This was done so that when we traced the hyperbola we got a nice smooth line. We cut a length of string we called C. Next we cut the straw to length R, where R is any length we want, as long at it is shorter than the string, but longer than the distance from a focus to the nearest vertex. We threaded the string through the straw and pinned it down where the two foci were chosen, and labeled the foci F_1 and F_2. In our device we chose to let the straw pivot around F_2. Lastly we used a pencil to keep the string taught by pushing the pencil inward along the outer side of the straw. We slid the pencil along the string where it left the surface of the straw and we got a hyperbola.



The Device for Drawing Hyperbolas

We have to prove that this device works, that it actually draws a hyperbola and not just something that closely resembles one. The two lengths C and R, along with the two foci, are the constants we used.

Choose a point P on the hyperbola. We let PF_1 be the distance from the point on the hyperbola to F_1 and PF_2 be the same for F_2.

$$C = R + (R - PF_1) + (PF_2)$$

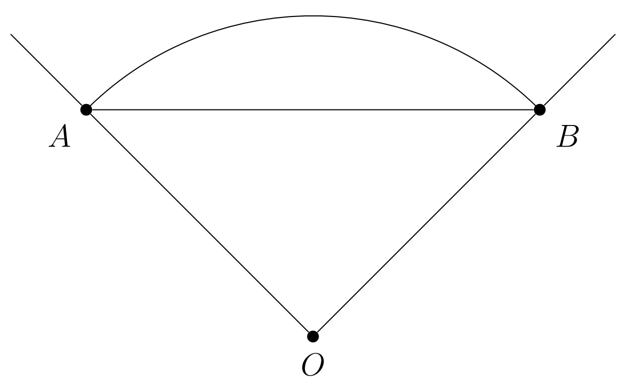
And so we see that:

$$PF_2 - PF_1 = C - 2R$$

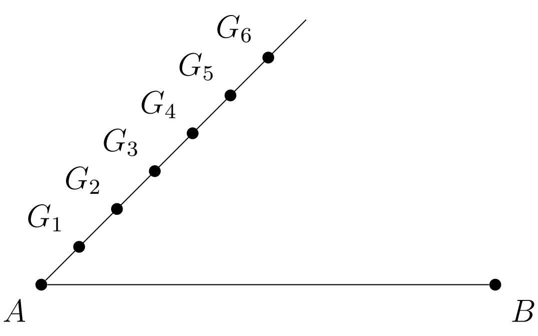
This shows us that for any point on the hyperbola, the difference between the distances to each focus is constant.

With out device made and the math done, we finished our quest. We spent the rest of the day drawing hyperbolas to our hearts content.

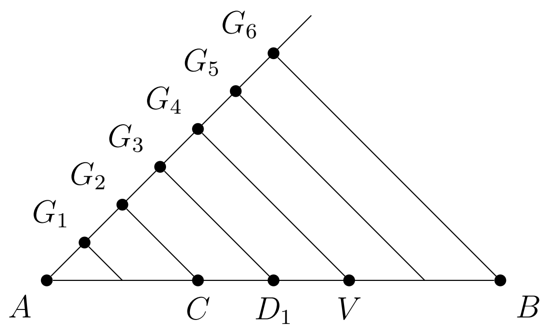
2.4. **Using the Device to Complete the Trisection.** Given an angle $\angle O$, mark a point A on on the the given rays. Draw a circle, centered at O with radius $\overline{OA}$. Mark the intersection on the second ray B. Draw the segment $\overline{AB}$.



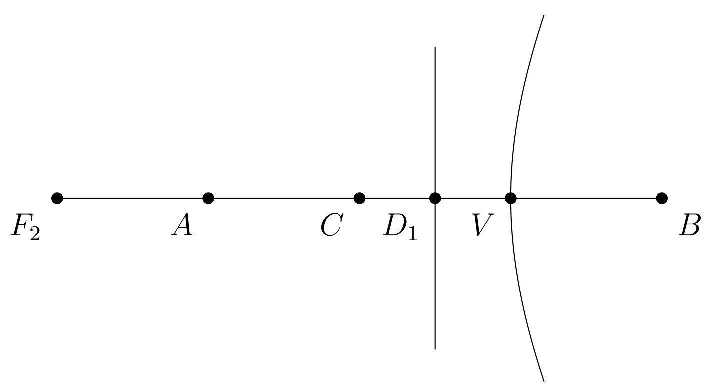
The next section focuses on the segment $\overline{AB}$. Divide the segment $\overline{AB}$ into 6 equal parts. To do this, we pick a point G_1, not on $\overline{AB}$, and draw the ray $\overline{AG_1}$. Mark points G_2, G_3, G_4, G_5, and G_6 on the ray such that $\overline{AG_1} = \overline{G_1G_2} = \overline{G_2G_3} = \overline{G_3G_4} = \overline{G_4G_5} = \overline{G_5G_6}$.



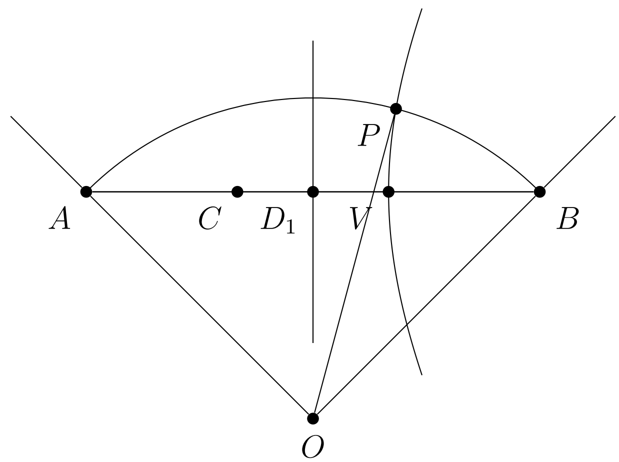
Draw $\overline{G_6B}$. Draw lines through G_1, G_2, G_3, G_4 and G_5 parallel to $\overline{G_6B}$. Each intersection produces equal length line segments on $\overline{AB}$. Mark each intersection as shown, and treat each segment as a unit length of one.



Extend $\overline{AB}$ past A a length of 2 units as shown below. Mark this point F_2. Construct a line perpendicular to AB through the point D_1. Using F_2 and B as the foci and V as the vertex, use the device to construct a hyperbola, called h. Since the distance from the the center, C, to F_1 is 4 units and the distance C to the vertex, V, is 2 units, the hyperbola has eccentricity of 2 as required.



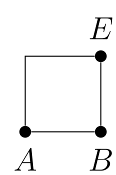
Mark the intersection point between the hyperbola, h, and the circle $\widehat{OA}$ as P. Draw the segment $\overline{OP}$. The angle $\angle POB$ trisects $\angle AOB$.



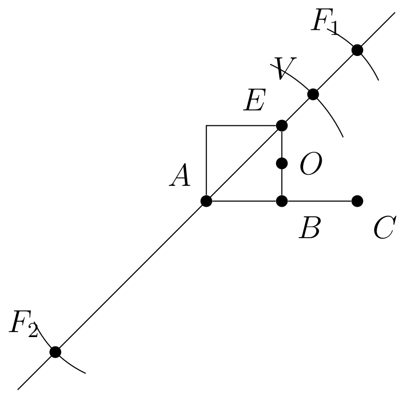
3. Constructing $\sqrt[3]{2}$

3.1. **The Construction.** We will begin this time with the construction first. Start with a given unit length of $\overline{AB}$.

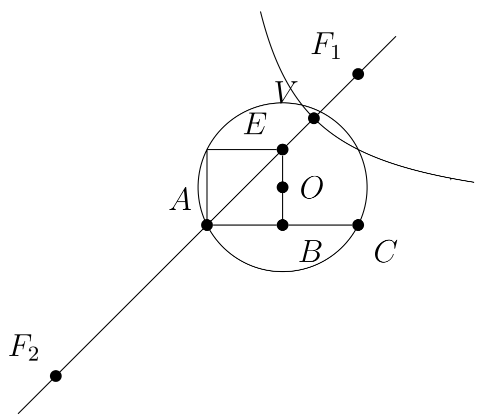
Construct a square with side $\overline{AB}$ and mark the point shown.



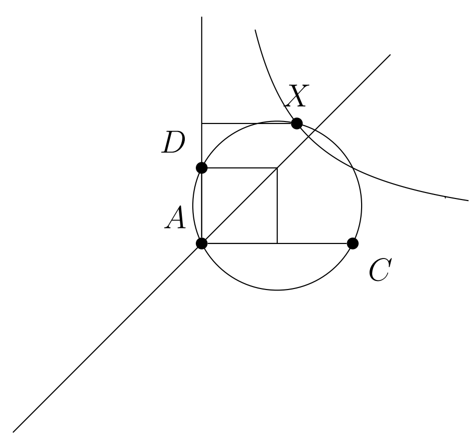
Draw a line l through the points A and E. Extend line $\overline{AB}$ past B a unit length of $\overline{AB}$, such that $\overline{AB} = \overline{BC}$. Draw a circle, centered at A, with radius $\overline{AC}$ and mark the intersections on the line l as V as shown. Draw the circle centered at E with radius $\overline{AE}$ and mark the intersection on l as F_1. Now draw the circle centered at A with radius $\overline{AF_1}$ and mark that intersection on l as F_2. Finally, bisect the segment $\overline{EB}$ and mark the point O.



Draw a circle centered at O with a radius of $\overline{OA}$. Using the device, draw a hyperbola with foci F_1 and F_2 and vertex V.



The circle intersects the hyperbola twice. Mark the leftmost intersection X and draw a perpendicular line from $\overline{AC}$ to X. This segment has length $\sqrt[3]{2}$.



3.2. **Proof of the Construction.** We can easily prove that the above construction is valid if we translate the above into Cartesian coordinates. If we allow the point A to be treated as the origin of the x - y plane and B be the point $(1, 0)$, we can write the equations of the circle and hyperbola. The circle is centered at 1 unit to the right and $\frac{1}{2}$ units up, giving it a center of $(1, \frac{1}{2})$ and a radius of $\sqrt{\frac{5}{4}}$. This gives the circle the equation:

$$(x-1)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{5}{4}.$$

The hyperbola, being rectangular with vertex $(\sqrt{2}, \sqrt{2})$, has the equation $xy = 2$. If we solve for y, we get $y = 2/x$. Substituting this expression into the circle's equation and solving for x yields the following:

$$\begin{aligned}
(x-1)^2 + \left(\frac{2}{x} - \frac{1}{2}\right)^2 &= \frac{5}{4} \\
x^2 - 2x + 1 + \frac{4}{x^2} - \frac{2}{x} + \frac{1}{4} &= \frac{5}{4} \\
x^2 - 2x + \frac{4}{x^2} - \frac{2}{x} + \frac{5}{4} &= \frac{5}{4} \\
x^2 - 2x + \frac{4}{x^2} - \frac{2}{x} &= 0 \\
x^4 - 2x^3 - 2x + 4 &= 0 \\
x^4 - 2x - 2x^3 + 4 &= 0 \\
x(x^3 - 2) - 2(x^3 - 2) &= 0 \\
(x^3 - 2)(x - 2) &= 0
\end{aligned}$$

From here we can see that both $x = \sqrt[3]{2}$ and $x = 2$ are solutions. This proves that the horizontal distance from the y-axis to the point X is $\sqrt[3]{2}$.

4. Acknowledgments

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References

[1] Heath, T., *A History of Greek Mathematics*, Dover Publications, New York, 1981.

[2] Apostol, T.M. and Mnatsakanian, M.N., *Ellipse to Hyperbola: With This String I Thee Wed*, Mathematics Magazine 84 (2011) 83-97.

[3] `http://en.wikipedia.org/wiki/Compass_equivalence_theorem` [Accessed 07/07/2011].

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