Decomposing a Symmetric Matrix by Way of Its Exponential Function

Maliha Bahri, William Cole,
Brad Johnston, Madeline Mayes

The Exponential Function

$$e^{at} = 1 + at + \frac{a^2t^2}{2!} + \frac{a^3t^3}{3!} + \frac{a^4t^4}{4!} + \dots$$

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \frac{A^4t^4}{4!} + \dots$$

Properties

1. $e^{A(0)} = I$
2. $D(e^{At}) = Ae^{At}$, where D is the differentiation with respect to t
3. For all t, e^{At} is invertible so, ${e^{At}}^{-1} = e^{-At}$
4. When $D(e^{At}) = Ae^{At}$ with the initial condition $e^{A(0)} = 0$ then there is a unique solution which is the zero vector
5. $e^{A}e^{B} = e^{A+B}$ when $AB = BA$

Linear Algebra Definitions

- Eigenvalue
- Eigenvector
- Eigenspace
- Image Space
- Span

The Laplace Transform Method

$$e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$$

$$C_A(s) = \det(sI - A)$$

$$\mathcal{L}\{e^{At}\} = (sI - A)^{-1}$$

$$\mathcal{L}\{e^{At}\} = \frac{1}{C_A(s)} B$$

An Example

$$A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$$

$$sI - A = \begin{pmatrix} s-4 & -2 & -2 \\ -2 & s-4 & -2 \\ -2 & -2 & s-4 \end{pmatrix}$$

$$\begin{aligned} C_A(s) &= s^3 - 12s^2 + 36s - 32 \\ (s-2)^2(s-8) &= 0 \\ s &= 2, 8 \end{aligned}$$

Using the Laplace Transform

$$\mathcal{L}\{e^{At}\} = \frac{1}{s^3 - 12s^2 + 36s - 32}B$$

$$\mathcal{L}\{e^{At}\} = \frac{1}{(s-2)^2(s-8)}B$$

$$\mathcal{L}\{e^{At}\} = \frac{M}{s-2} + \frac{N}{(s-2)^2} + \frac{P}{s-8}$$

$$e^{At} = Me^{2t} + Nte^{2t} + Pe^{8t}$$

Equations for Coefficient Matrices

$$
\begin{aligned}
e^{At} &= Me^{2t} + Nte^{2t} + Pe^{8t} \\
Ae^{At} &= 2Me^{2t} + N(e^{2t} + 2te^{2t}) + 8Pe^{8t} \\
A^2e^{At} &= 4Me^{2t} + N(4e^{2t} + 4te^{2t}) + 64Pe^{8t}
\end{aligned}
$$

When t=0

$$
\begin{aligned}
I &= M + P \\
A &= 2M + N + 8P \\
A^2 &= 4M + 4N + 64P
\end{aligned}
$$

Solving for M, N and P

$$M = I - \frac{A^2 - 4A + 4I}{36}$$

$$N = \frac{A^2 - 10A + 16I}{-6}$$

$$P = \frac{A^2 - 4A + 4I}{36}$$

$$M = \begin{pmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{pmatrix}$$

$$N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$P = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$$

Projections

If an operator, like matrix M, is squared and equals itself then we say that M is a **projection**.

$$M^2 = \begin{pmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$$

Spectral Decomposition

$$A = 2M + N + 8P$$

$$A = 2M + 8P$$

$$A = \sum_i \lambda_i M_i$$

Eigenspace of λ_i

$$E_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$E_8 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Image Space of the Projections

$$span(M) = \left\{ \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \right\}$$

$$span(P) = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

The Point

Let A be a symmetric matrix, with k distinct eigenvalues λ_i. There exists matrices P_i so that

$$A = \sum \lambda_i P_i$$

and P_i holds the following properties:

1. $I = \sum P_i$
2. $P_i^2 = P_i$
3. $P_i P_j = 0,\ i \neq j.$

Moreover, $Im(P_i) = E_i$ where E_i is the eigenspace of λ_i.