

Decomposing a Symmetric Matrix by Way of Its Exponential Function

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The Exponential Function

$$e^{at} = 1 + at + \frac{a^2 t^2}{2!} + \frac{a^3 t^3}{3!} + \frac{a^4 t^4}{4!} + \dots$$

$$e^{At} = I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \frac{A^4 t^4}{4!} + \dots$$



Properties

1. $e^{A(0)} = I$
2. $D(e^{At}) = Ae^{At}$, where D is the differentiation with respect to t
3. For all t , e^{At} is invertible so, $e^{At^{-1}} = e^{-At}$
4. When $D(e^{At}) = Ae^{At}$ with the initial condition $e^{A(0)} = 0$ then there is a unique solution which is the zero vector
5. $e^A e^B = e^{A+B}$ when $AB = BA$

Linear Algebra Definitions

- Eigenvalue
- Eigenvector
- Eigenspace
- Image Space
- Span



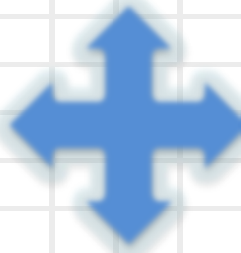
The Laplace Transform Method

$$e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$$

$$C_A(s) = \det(sI - A)$$

$$\mathcal{L}\{e^{At}\} = (sI - A)^{-1}$$

$$\mathcal{L}\{e^{At}\} = \frac{1}{C_A(s)} B$$



An Example

$$A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$$

$$sI - A = \begin{pmatrix} s-4 & -2 & -2 \\ -2 & s-4 & -2 \\ -2 & -2 & s-4 \end{pmatrix}$$

$$C_A(s) = s^3 - 12s^2 + 36s - 32$$

$$(s-2)^2(s-8) = 0$$

$$s = 2, 8$$



Using the Laplace Transform

$$\mathcal{L}\{e^{At}\} = \frac{1}{s^3 - 12s^2 + 36s - 32} B$$

$$\mathcal{L}\{e^{At}\} = \frac{1}{(s - 2)^2(s - 8)} B$$

$$\mathcal{L}\{e^{At}\} = \frac{M}{s - 2} + \frac{N}{(s - 2)^2} + \frac{P}{s - 8}$$

$$e^{At} = Me^{2t} + Nte^{2t} + Pe^{8t}$$



Equations for Coefficient Matrices

$$e^{At} = Me^{2t} + Nte^{2t} + Pe^{8t}$$

$$Ae^{At} = 2Me^{2t} + N(e^{2t} + 2te^{2t}) + 8Pe^{8t}$$

$$A^2e^{At} = 4Me^{2t} + N(4e^{2t} + 4te^{2t}) + 64Pe^{8t}$$






When $t=0$

$$I = M + P$$

$$A = 2M + N + 8P$$

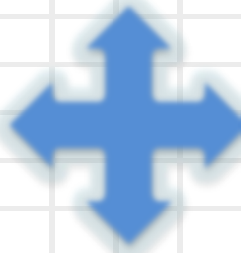
$$A^2 = 4M + 4N + 64P$$


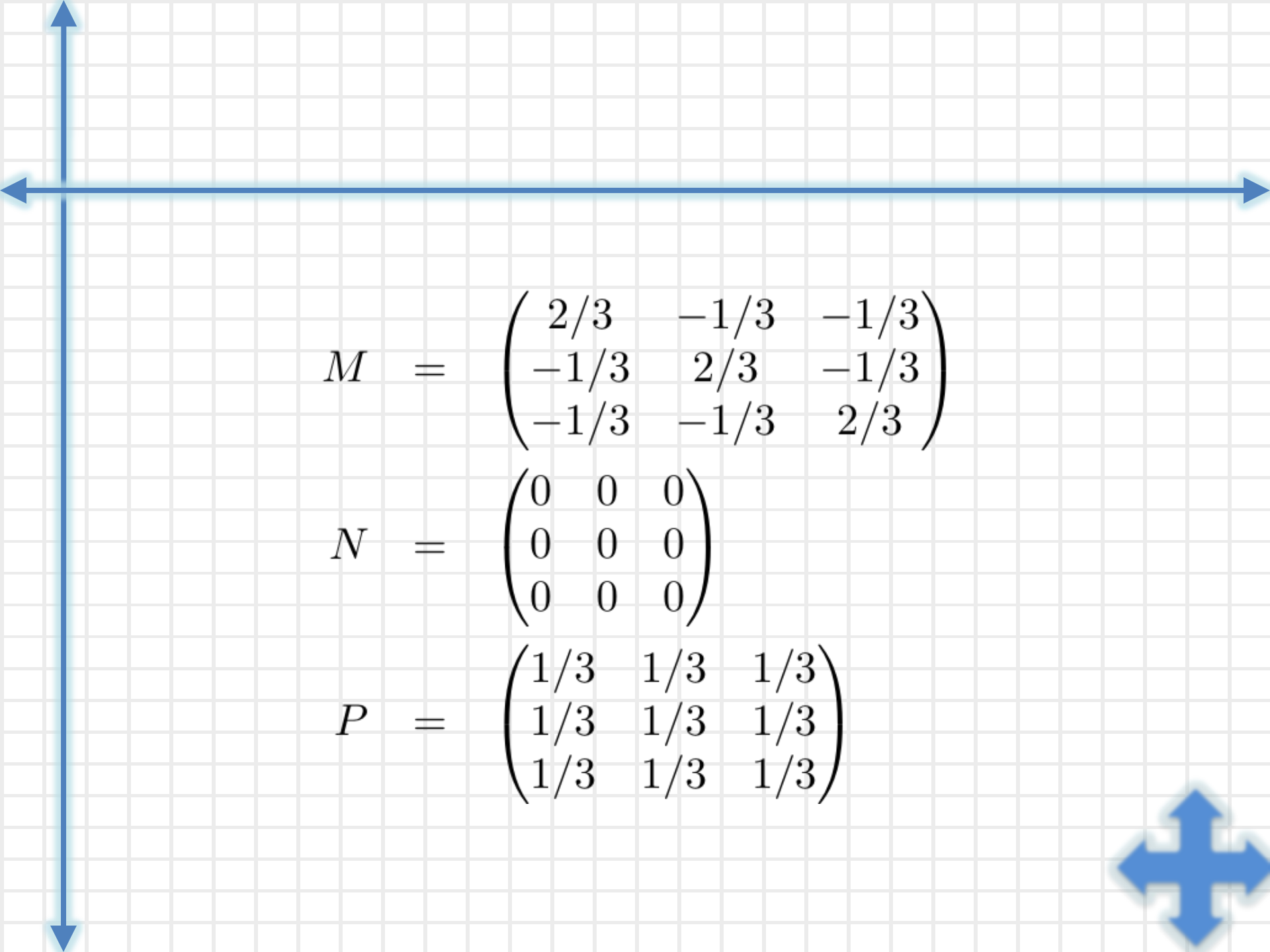
Solving for M, N and P

$$M = I - \frac{A^2 - 4A + 4I}{36}$$


$$N = \frac{A^2 - 10A + 16I}{-6}$$

$$P = \frac{A^2 - 4A + 4I}{36}$$




$$M = \begin{pmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{pmatrix}$$

$$N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$P = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$$


Projections

If an operator, like matrix M , is squared and equals itself then we say that M is a **projection**.

$$M^2 = \begin{pmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$$



Spectral Decomposition

$$A = 2M + N + 8P$$

$$A = 2M + 8P$$

$$A = \sum_i \lambda_i M_i$$



Eigenspace of λ_i

$$E_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$E_8 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$



Image Space of the Projections

$$\text{span}(M) = \left\{ \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \right\}$$

$$\text{span}(P) = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$



The Point

Let A be a symmetric matrix, with k distinct eigenvalues λ_i . There exists matrices P_i so that

$$A = \sum \lambda_i P_i$$

and P_i holds the following properties:

1. $I = \sum P_i$
2. $P_i^2 = P_i$
3. $P_i P_j = 0, i \neq j.$

Moreover, $Im(P_i) = E_i$ where E_i is the eigenspace of λ_i .

