DECOMPOSING A SYMMETRIC MATRIX BY WAY OF ITS EXPONENTIAL FUNCTION

MALIHA BAHRI, WILLIAM COLE, BRAD JOHNSTON,
AND MADELINE MAYES

Abstract. It is well known that one can develop the spectral theorem to decompose a matrix. In this article, we develop this structure theorem through an uncommon method by examining the matrix exponential of a symmetric matrix, which is explicitly computable by the Laplace transform.

1. e^{At}, Its Properties, and Concepts in Linear Algebra

Often we try to extend ideas from a familiar space into an unfamiliar one. With this train of thought in mind, consider the Taylor expansion of the real exponential.

$$e^{at} = 1 + at + \frac{a^2t^2}{2!} + \frac{a^3t^3}{3!} + \frac{a^4t^4}{4!} + \dots$$

Based on this Taylor expansion, one commonly defines the matrix exponential for a matrix A to be

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \frac{A^4t^4}{4!} + \dots$$

Furthermore, e^{At} has the following properties:

(1) $e^{A(0)} = I$,
(2) $D(e^{At}) = Ae^{At}$, where D is the differentiation with respect to t,
(3) for all t, e^{At} is invertible so, $e^{At^{-1}} = e^{-At}$,
(4) when $D(e^{At}) = Ae^{At}$ with the initial condition $e^{A(0)} = 0$ then there is a unique solution which is the zero vector, and
(5) $e^Ae^B = e^{A+B}$ when $AB = BA$.

Before we begin, here are some important definitions to help understand our paper.

Definition 1.1. A scalar λ is called an *eigenvalue* of the matrix A if there exists a non-zero vector v such that $Av = \lambda v$.

Date: 10 July 2009.

Definition 1.2. A non-zero vector v is called an *eigenvector* of the matrix A if there exists a scalar λ such that $Av = \lambda v$.

Definition 1.3. The *eigenspace* E_i for a matrix A for its eigenvalue λ_i is the set of all vectors V such that there exists a $v \in V$ where $Av = \lambda_i v$.

Definition 1.4. The *image space* of a matrix A is the set of all $W \subseteq V$ and there exists a $w \in W$ and $v \in V$ such that $A(v) = w$ and we denote this as $Im(A)$.

Definition 1.5. For any subset v of V the set of all linear combinations of vectors in v is called the set *spanned* by v.

2. Cayley-Hamilton Theorem and e^{At}

The Cayley-Hamilton Theorem states that if $C_A(s)$ is the characteristic polynomial of a matrix A, then $C_A(A) = 0$. So using the second property of e^{At} we have

$$\begin{aligned} De^{At} &= Ae^{At} \\ C_A(De^{At}) &= C_A(Ae^{At}). \end{aligned}$$

We can factor out e^{At} since it is not a parameter of C_A. Thus,

$$\begin{aligned} C_A(D)e^{At} &= C_A(A)e^{At} \\ C_A(D)e^{At} &= 0. \end{aligned}$$

Since we have $e^{At} \neq 0$ then $C_A(D) = 0$. So, when we solve this differential equation for each entry of e^{At}, $\beta_{i,j}$, we have the form:

$$\beta_{i,j} = \sum_{\mu=1}^{k} \sum_{v=0}^{m_\mu - 1} t^v e^{\lambda_\mu t} c_{\mu,v}.$$

We will have a matrix coefficient of $c_{\mu,v}$'s for each $e^{\lambda t}$, where λ is also an eigenvalue of A, in other words,

$$e^{At} = \sum_{\mu=1}^{k} \sum_{v=0}^{m_\mu - 1} t^v e^{\lambda_\mu t} M_{\mu,v} \tag{2.1}$$

where we call the coefficient matrices M. If an eigenvalue has a multiplicity of $v \geq 2$, then its $M_{\mu,v}$ is multiplied by a factor of t^v for each eigenvalue λ_μ [2].

3. Laplace Transform Method and e^{At}

Theorem 3.1. *For any square matrix A and $\mathcal{L}$ the Laplace transform,*

$$e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}.$$

We can also obtain the summation from 2.1 if we look at Theorem 3.1 using the Laplace transform. If we take the Laplace transform of both sides we have

$$\mathcal{L}\{e^{At}\} = (sI - A)^{-1}.$$

Since all of the elements in the matrix have a factor of the characteristic polynomial $C_A(s)$, we can say,

$$\mathcal{L}\{e^{At}\} = \frac{1}{C_A(s)}B$$

where B is a matrix of the remaining factor. If $C_A(s)$ is factorable, we can rewrite $\mathcal{L}\{e^{At}\}$ as the sum

$$\mathcal{L}\{e^{At}\} = \sum_{\mu=1}^{k}\sum_{v=0}^{m_\mu - 1} t^v \frac{1}{s - \lambda_\mu} M_{\mu,v},$$

where $s - \lambda_\mu$ is a factor of $C_A(s)$ and M is a coefficient matrix from decomposing B. When we take the inverse Laplace transform of both sides, we again have:

$$e^{At} = \sum_{\mu=1}^{k}\sum_{v=0}^{m_\mu - 1} t^v e^{\lambda_\mu t} M_{\mu,v}. \tag{3.1}$$

4. A 3x3 Example

The following example decomposes a 3 x 3 symmetric matrix.
Let

$$A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}.$$

First, we find the characteristic polyonmial by the formula
$C_A(\lambda) = det(\lambda I - A)$. We get

$$C_A(\lambda) = (\lambda - 8)(\lambda - 2)^2.$$

If we set the characteristic polynomial equal to 0, we find the eigenvalues are $\lambda_1 = 2$ with multiplicity 2 and $\lambda_2 = 8$. Using the equation from 2.1 and 3.1, we rewrite $e^{At} = Me^{\lambda_1} + Nte^{\lambda_1} + Pe^{\lambda_2}$. Now, we can plug in our eigenvalues to get

$$e^{At} = Me^{2t} + Nte^{2t} + Pe^{8t}.$$

Using the properties of e^{At}, we can derive the 2^{nd} and 3^{rd} derivatives to find

$$\begin{aligned} Ae^{At} &= 2Me^{2t} + N(e^{2t} + 2te^{2t}) + 8Pe^{8t} \\ A^2e^{At} &= 4Me^{2t} + N(4e^{2t} + 4te^{2t}) + 64Pe^{8t}. \end{aligned}$$

If we evalute these equations at t=0, then we get the following system of equations:

$$\begin{aligned} I &= M + P \\ A &= 2M + N + 8P \\ A^2 &= 4M + 4N + 64P. \end{aligned}$$

By solving this system of equations we find the following matrices:

$$M = \begin{pmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{pmatrix},$$
$$N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$P = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}.$$

Definition 4.1. If an operator, like matrix M, is squared and equals itself then we say that M is a *projection*.

So we square these matrices to find

$$M^2 = \begin{pmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{pmatrix},$$
$$P^2 = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}.$$

We see that $M = M^2$ and $P = P^2$ which shows that M and P are projections.

It is now clear that A is decomposed into three matrices. However, we can omit N because it is the zero matrix and we write:

$$\begin{aligned} A &= 2M + 8P \\ &= 2\begin{pmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{pmatrix} + 8\begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} \end{aligned}$$

We now realize that we have rewritten A as the eigenvalues multiplied by the projection matrices. This is known as a *spectral decomposition.*

Another important thing to note is that these matrices are projections onto their corresponding eigenspaces. In other words,

$$\begin{aligned} Im\{M\} &= \text{eigenspace of } 2 \\ Im\{P\} &= \text{eigenspace of } 8. \end{aligned}$$

First we calculate the image space, which is equal to the span of the columns.

So the image space of M is

$$span\left\{\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}\right\}.$$

Next we find the basis of the eigenspace of $\lambda = 2$, and this is

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

We use the same method for P to find that the image space of P, is

$$span\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\},$$

and the basis of the eigenspace of $\lambda = 8$ is

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

So, it is clear that the image space of P is equal to the eigenspace of $\lambda = 8$.

To show that the image space of M is equal to the eigenspace of 2, we write the eigenspace as a linear combination of the two bases.

$$2\begin{pmatrix}1\\0\\-1\end{pmatrix} - 1\begin{pmatrix}0\\1\\-1\end{pmatrix} = \begin{pmatrix}2\\-1\\-1\end{pmatrix}$$

$$-1\begin{pmatrix}1\\0\\-1\end{pmatrix} + 2\begin{pmatrix}0\\1\\-1\end{pmatrix} = \begin{pmatrix}-1\\2\\-1\end{pmatrix}$$

From these results, we can conclude that the image spaces of M and P are equal to the eigenspaces.

5. Looking at 2x2 Matrices

When given a 2 x 2 matrix A, the characteristic polynomial is

$$C_A(\lambda) = (\lambda - \alpha_{11})(\lambda - \alpha_{22}) - (\alpha_{12})(\alpha_{22}).$$

Setting $C_A(\lambda) = 0$, we find the eigenvalues of A. Because $n = 2$, $C_A(\lambda)$ will be a quadratic equation when set equal to zero. Solving this quadratic equation yields one or two distinct values in the complex reals. Examining a general 2 x 2 matrix A,

$$A = \begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix}.$$

The first case that we are presented with is there being one eigenvalue with multiplicity 2. The properties of e^{At} and equation 3.1 state

$$e^{At} = Me^{\lambda t} + Nte^{\lambda t}$$

and differentiating e^{At} gives us

$$Ae^{At} = \lambda Me^{\lambda t} + N(e^{\lambda t} + \lambda te^{\lambda t}).$$

Evaluating these equations at t=0 produces the following system of equations:

$$\begin{aligned} I &= M \\ A &= \lambda I + N \end{aligned}$$

We see from the first equation M is the identity matrix and $N = A - \lambda I$. Futhermore, N is a nilpotent matrix and M is a projection.

Definition 5.1. A *nilpotent* matrix M is a matrix where there exists a u such that $M^u = 0$.

The second case that could arise is when we have two distinct eigenvalues associated with matrix A. We can use the properties of $e^A(t)$ to find the first derivative.

$$Ae^{At} = \lambda_1 M e^{\lambda_1 t} + \lambda_2 N e^{\lambda_2 t}$$

When we evaluate both of these equations at $t = 0$, the following system of equations result

$$\begin{aligned} I &= M + N \\ A &= \lambda_1 M + \lambda_2 N. \end{aligned}$$

From here we can find each entry of the coefficient matrices M and N by finding each entry of the matrix. Since the identity matrix equals

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

From the first equation we find that

$$\begin{aligned} M_{11} + N_{11} &= 1 \\ M_{12} + N_{12} &= 0 \\ M_{21} + N_{21} &= 0 \\ M_{22} + N_{22} &= 1. \end{aligned}$$

We also know that

$$\begin{aligned} \lambda_1 M_{11} + \lambda_2 N_{11} &= \alpha_{11} \\ \lambda_1 M_{12} + \lambda_2 N_{12} &= \alpha_{12} \\ \lambda_1 M_{21} + \lambda_2 N_{21} &= \alpha_{21} \\ \lambda_1 M_{21} + \lambda_2 N_{11} &= \alpha_{21}. \end{aligned}$$

Combining these equations we can solve for the coefficient matrices M and N.

$$M = \begin{pmatrix} \frac{\alpha_{11}-\lambda_2}{\lambda_1-\lambda_2} & \frac{\alpha_{12}}{\lambda_1-\lambda_2} \\ \frac{\alpha_{21}}{\lambda_1-\lambda_2} & \frac{\alpha_{22}-\lambda_2}{\lambda_1-\lambda_2} \end{pmatrix}.$$

$$N = \begin{pmatrix} \frac{1-(\alpha_{11}-\lambda_2)}{\lambda_1-\lambda_2} & \frac{-\alpha_{12}}{\lambda_1-\lambda_2} \\ \frac{-\alpha_{21}}{\lambda_1-\lambda_2} & \frac{1-(\alpha_{22}-\lambda_2)}{\lambda_1-\lambda_2} \end{pmatrix}.$$

Presented with a matrix A and computing the eigenvalues, we can easily find the coefficient matrices M and N. Next, squaring M and N will show the matrices are projections. Let

$$A = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix}.$$

The eigenvalues associated with this matrix are $\lambda_1 = 6$, and $\lambda_2 = -2$.

$$M = \begin{pmatrix} 3/8 & 3/8 \\ 5/8 & 5/8 \end{pmatrix}$$
$$N = \begin{pmatrix} 5/8 & -3/8 \\ -5/8 & 3/8 \end{pmatrix}.$$

Squaring M and N yields

$$M^2 = \begin{pmatrix} 3/8 & 3/8 \\ 5/8 & 5/8 \end{pmatrix}$$
$$N^2 = \begin{pmatrix} 5/8 & -3/8 \\ -5/8 & 3/8 \end{pmatrix}.$$

Thus, they are projections.

6. A Structure Theorem for Symmetric Matrices

Lemma 6.1. *Let N be a nilpotent matrix of order m. If N is symmetric then $N = 0$.*

Proof. for 6.1:
Let $w \in ker(N)$ and $v \notin ker(N)$.
Assume $N^P v \neq 0$ for some $p \in \mathbb{N}$.

$$0 = \langle N^{p-1}v, Nw \rangle = \langle N^P v, w \rangle$$

So $N^P \in ker(N)^\perp$ and $N^P v \neq 0$.
Thus, $N^P \notin ker(N)$ and $N^{p+1}v \neq 0$,
$N^P v \neq 0 \to N^{p+1} v \neq 0$
Therefore $Nv \neq 0$ implies $N^P v \neq 0$ for all $p \in \mathbb{N}$. But this is false since N is nilpotent. Thus $N = 0$.

□

Theorem 6.2. *Let A be a symmetric matrix, with k distinct eigenvalues λ_i. There exists matrices P_i so that*

$$A = \sum \lambda_i P_i$$

and P_i holds the following properties:

1. $I = \sum P_i$
2. $P_i^2 = P_i$
3. $P_i P_j = 0$, $i \neq j$.

Moreover, $Im(P_i) = E_i$ where E_i is the eigenspace of λ_i.

Proof. (i)
In order to prove that $I = \sum P_i$, recall 2.1,

$$e^{At} = \sum_{\mu=1}^{k} \sum_{v=0}^{m_\mu - 1} t^v e^{\lambda_\mu t} M_{\mu,v}.$$

If we set $t = 0$, it is clear that

$$I = \sum_{\mu=1}^{k} M_{\mu,v}.$$

(ii)
In order prove that $P_i^2 = P_i$ and that $P_i P_j = 0$ when $i \neq j$ we simplify 2.1 and let

$$M_{\mu,v} = 0 \text{ for } v \geq m_\mu,$$

so 2.1 becomes

$$e^{At} = \sum_{u=1}^{k} \sum_{v=0}^{n} t^v e^{\lambda_\mu t} M_{\mu,v}$$

where n is the highest order of the eigenvalues of A.

Thus for arbitrary constants r and s we have

$$e^{Ar} e^{As} = \sum_{\rho,\xi=1}^{k} \sum_{\sigma,\eta=0}^{n} r^\sigma s^\eta e^{\lambda_\rho r + \lambda_\xi s} M_{\rho,\sigma} M_{\xi,\eta}$$

$$e^{A(r+s)} = \sum_{\mu=1}^{k} \sum_{v=0}^{n} (r+s)^v e^{\lambda_\mu (r+s)} M_{\mu,v}.$$

Recall the Kronecker Delta

$$\delta_{ij} = 1 \text{ if and only if } i = j$$

We can rewrite our sum as

$$e^{A(r+s)} = \sum_{\rho,\xi=1}^{k} \sum_{v=0}^{n} (r+s)^v \delta_{\rho,\xi} e^{\lambda_\rho r + \lambda_\xi s} M_{\rho,v}.$$

By employing the Binomial Theorem, we once more can rewrite our sum,

$$e^{A(r+s)} = \sum_{\rho,\xi=1}^{k} \sum_{\sigma,\eta=0}^{n} \delta_{\rho,\xi} \binom{\sigma+\eta}{\sigma} r^\sigma s^\eta e^{\lambda_\rho r + \lambda_\xi s} M_{\rho,\sigma+\nu}.$$

Using the equality established in the exponential function's property (5) we can find the following equation

$$M_{\rho,\sigma} M_{\xi,\eta} = \delta_{\rho,\xi} \binom{\sigma+\eta}{\sigma} M_{\rho,\sigma+\eta}. \tag{6.1}$$

Now fix some index ρ and set $\rho = \xi$ and $\sigma = \eta = 0$. Then it is clear that $M_{\rho,0}$ is a projection matrix.

(iii)
It is also clear from (6.1) that $M_{\rho,\sigma}M_{\xi,\eta} = 0$ when $\rho \neq \xi$.

Finally, in order prove that $Im(P_i) = E_i$ where E_i is the eigenspace of λ_i we look at equation (6.1). Let ρ and ξ remain equal and set $\eta = 1$ and let σ be arbitrary to obtain

$$M_{\rho,\sigma+1} = (\sigma+1)^{-1}M_{\rho,\sigma}M_{\rho,1}.$$

From this it is easy to obtain the formula

$$M_{\rho,\sigma} = M^{\sigma}_{\rho,1}/\sigma!.$$

As defined before, $M_{\rho,m_\rho} = 0$, thus $M^{m_\rho}_{\rho,1} = 0$. So $M_{\rho,1}$ is a nilpotent matrix. Hence all of our other matrices in the summation can be expressed in terms of a projection and a nilpotent matrix.

At this point, it is advantageous to change our notation to something a bit more readable.

$$M_{\rho,0} = P_\rho$$
$$M_{\rho,1} = N_\rho$$

These matrices have some nice properties that can be derived from (6.1).

(1) $P_\mu P_v = \delta_{\mu v}P_v$
(2) $P_\mu N_v = N_v P_\mu = \delta_{\mu v}N_v$
(3) $N_\mu N_v = \delta_{\mu v}N_v^2$
(4) $N_\mu^{m_\mu} = 0$

From these equations, it is simple to see that

$$M_{\mu,v} = (1/v!)N_\mu^v P_\mu.$$

So our formula for e^{At} becomes,

$$e^{At} = \sum_{\mu=1}^{k}\sum_{v=0}^{m_\mu-1}(t^v/v!)e^{\lambda_\mu t}N_\mu^v P_\mu.$$

We set $t = 0$ to see a decomposition of A:

$$A = \sum_{\mu=1}^{k}(\lambda_\mu I + N_\mu)P_\mu = \sum_{\mu=1}^{k}\lambda_\mu P_\mu + N_\mu.$$

It is easy to show that for a symmetric matrix A, the coefficient matrices are also symmetric. Recall the previous lemma 6.1.

Hence, we are left with the following decomposition

$$A = \sum_{i=1}^{k}\lambda_i P_i. \tag{6.2}$$

Suppose $w \in Im(P_i)$. Thus, there exists a $v \in V$ such that $P_i(v) = w$. Apply A to both sides of this equation to get

$$A(P_i(v)) = A(w).$$

Recall (6.2) and we see that

$$A(P_i(v)) = \lambda_1 P_1(P_i(v)) + \ldots + \lambda_i P_i(P_i(v)) + \ldots + \lambda_k P_k(P_i(v))$$

$$A(P_i(v)) = \lambda_i P_i^2(v) = \lambda_i P_i(v) = \lambda_i w.$$

Thus,

$$\lambda_i w = A(w).$$

Thus we have proved that $w \in E_i$. Hence, $Im(P_i) \subseteq E_i$.

Now suppose $w \in E_i$. This gives us the equation

$$A(w) = \lambda_i w.$$

Recall (6.2) and we notice that

$$\lambda_1 P_1(w) + \ldots + \lambda_k P_k(w) = \lambda_i w$$

$$\lambda_1 P_1(w) + \ldots + \lambda_i P_i(w) - \lambda_i w + \ldots + \lambda_k P_k(w) = 0.$$

Now we pick a $j \neq i$.

$$\lambda_1 P_j(P_1(w)) + \ldots + \lambda_i P_j(P_i(w)) - \lambda_i P_j(w) + \ldots + \lambda_k P_j(P_k(w)) = M_j(0)$$

Since $P_j P_i = \delta_{i,j} P_i$, we have

$$\lambda_j P_j^2(w) - \lambda_i P_j(w) = 0$$

$$\lambda_j P_j(w) = \lambda_i P_j(w).$$

But since we implicitly picked $\lambda_i \neq \lambda_j$, we can conclude that,

$$P_j(w) = 0 \text{ for all } j \neq i$$

$$\lambda_i P_i(w) - \lambda_i w = 0 \Rightarrow P_i(w) = w.$$

Thus $w \in Im(P_i)$. Thus $E_i \subseteq Im(P_i)$. Combined with the above result we can conclude that $Im(P_i) = E_i$. ☐

7. Conclusion

We see that the coefficient matrices that we derive using the Laplace transform of the exponential matrix are projections. Moreover, these matrices are also projections of the original symmetric matrix. Therefore, we develop a different method to obtain the spectral decomposition. The image spaces of these projections equal the eigenspaces of their corresponding eigenvalues. It is also interesting to note, that we do not find any nilpotent matrices in a symmetric matrix's decomposition besides the zero matrix. The exponential function method to develop the spectral theorem can be extended to any matrix with complex entries. We see this done in Dr. Ziebur's paper [2].

Acknowledgements

The authors thank Dr. Mark Davidson for introducing different ways of applying the Laplace transform and special thanks to Adam Cross for the motivating discussions concerning this method.

References

[1] Nering, E. D., *Linear Algebra and Matrix Theory*, John Wiley and Sons, Inc., New York, 1963, pp. 12, 19, 91, 94.

[2] Ziebur, A. D., On Determining the Structure of A by Analyzing e^{At} in *SIAM Review Vol. 12 No. 1*, Jan 1970, pp. 98-102.

Mathematics Department, Louisiana State University, Baton Rouge, Louisiana

E-mail address: mbahri1@tigers.lsu.edu

Mathematics Department, The University of Mississippi, University, Mississippi

E-mail address: wacole@olemiss.edu

Mathematics Department, Louisiana State University, Baton Rouge, Louisiana

E-mail address: bjohn15@tigers.lsu.edu

Mathematics Department, Louisiana State University, Baton Rouge, Louisiana

E-mail address: mmayes2@tigers.lsu.edu