A Course on the Yosida Theorem

Classical & Pointfree Versions & Applications

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Lecture 11

Categories of Representations

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Review: Localic Yosida

Suppose A is an archimedean ℓ-group, and $e \in A^+$.

(i) Let $y : A^+ \to \mathcal{Y}A$ be the set map universal for the following relations ($a, b \in A^+$):

(l_1) $y(0) = \bot$,
(l_2) $y(a \wedge b) = y(a) \wedge y(b)$,
(l_3) $y(a + b) = y(a) \vee y(b)$,
(l_4) $y(a \vee b) = y(a) \vee y(b)$.
(Y) if $\{a_i\}_{i=0}^{\infty} \subseteq A^+$ and $a_i \uparrow_b a$, then $y(a) = \bigvee_{i=1}^{\infty} y(a_i)$.

Then $\mathcal{Y}A$ is order-isomorphic to the augmentation of the frame of archimedean kernels of A.

(ii) Let $y_e : A^+ \to \mathcal{Y}_e A$ be the quotient of $\mathcal{Y}A$ obtained by adding the relation:

(U_e) $y_e(e) = \top$.

Then $\mathcal{Y}_e A$ regular Lindelöf.

(iii) Let $\Phi_e : A \to \mathcal{R}\,\mathcal{Y}_e A$ be defined by

$$\Phi_e(a)(p, q) = y_e\left((a - pe)^+ \wedge (qe - a)^+\right), \quad p, q \in \mathbb{Q}.$$

Then Φ_e is an ℓ-homomorphism with kernel $e^{\perp}$.

Funtoriality of $\mathcal{Y}$

Suppose $\beta : A \to B$ is an ℓ-homomorphism of archimedean ℓ-groups.

- There is a frame morphism $\mathcal{Y}(\beta) : \mathcal{Y}(A) \to \mathcal{Y}(B); y(a) \mapsto y(\beta a)$.
- For any $e \in A^+$, there is $\mathcal{Y}_e(\beta)$ making the following diagram commute:

$$\begin{array}{ccc}
\mathcal{Y}(A) & \xrightarrow{\mathcal{Y}(\beta)} & \mathcal{Y}(B) \\
{\scriptstyle _\wedge y(e)}\downarrow & & \downarrow{\scriptstyle _\wedge y(\beta e)} \\
\mathcal{Y}_e(A) & \xrightarrow{\mathcal{Y}_e(\beta)} & \mathcal{Y}_{\beta e}(B)
\end{array}$$

(Here, we identify $\mathcal{Y}_e A$ with $[\bot, y(e)] \subseteq \mathcal{Y}A$.)

- For any $e \in A^+$ and $a \in A$, $\Phi_{\beta e}(\beta a) = \mathcal{Y}_e(\beta) \circ \Phi_e(a)$:

$$\begin{array}{ccc}
\mathcal{Y}_e(A) & \xrightarrow{\mathcal{Y}_e(\beta)} & \mathcal{Y}_{\beta e}(B) \\
{\scriptstyle \Phi_e(a)}\nwarrow & & \nearrow{\scriptstyle \Phi_{\beta e}(\beta a)} \\
& \mathcal{R} &
\end{array}$$

Review: Change of Unit

The image of A under the map Φ_e is denoted by $\Phi(A, e)$. By definition, there is a containment $\Phi(A, e) \subseteq \mathcal{RY}(A, e)$. Both these ℓ-groups are canonically **W**-objects: the unit in $\mathcal{RY}(A, e)$ is the (localic) constant function 1, which is $\Phi_e(e)$. Thus, the containment is a **W**-morphism.

We have seen that if $y(f) \leqslant y(e)$, then there is a frame map $\pi^e_f : \mathcal{Y}(A, e) \to \mathcal{Y}(A, f)$ and an ℓ-homomorphism (actually, an ℓ-orthomorphism):

$$\rho^e_f : \Phi(A, e) \to \Phi(A, f)$$
$$\rho^e_f(\phi) = \Phi_f(e) \cdot (\pi^e_f \circ \phi).$$

This map is "localic restriction," followed by multiplication by $\Phi_f(e)$. It is a surjection, and hence a **W**-morphism if we take $\pi^e_f \circ \Phi_e(e)$ as the weak unit of $\Phi(A, f)$. But $\pi^e_f \circ \Phi_e(e)$ is *not* in general equal to $\Phi_f(f)$, the canonical choice of weak unit in $\Phi(A, f)$. Last lecture, we proved:

Change of Unit Proposition. $\rho^e_f(\Phi_e(a)) = \Phi_f(a)$.

Example (7^{th}-grade proportional reasoning). Suppose A is the set of real-valued functions on the 2-point space. We write (c, d) to mean the function that has value c at the first point and d at the second. Let $e := (3, 4)$ and $f := (2, 0)$. Then $\Phi_e(c, d) = (c/3, d/4)$ and $\Phi_f(c, d) = (c/2)$.

$$\rho^e_f(\Phi_e(c, d)) = \Phi_f(e) \cdot (\pi^e_f \circ (c/3, d/4)) = (3/2) \cdot (c/3) = (c/2) = \Phi_f(c, d).$$

Comment. The Yosida theorem produces, for each $e \in A$, a representation $\Phi(A, e) \subseteq \mathcal{Y}(A, e)$. The discussion above tells us how the different representations, as we let e vary, relate to one another.

The Category **RL**

Definition. **RL** denotes the category described as follows:

(*a*) **RL**-objects are pairs $(\mathcal{O}, A)$, where:

- (*i*) $\mathcal{O}$ is a regular Lindelöf frame, and
- (*ii*) A is a sub-ℓ-group of $\mathcal{R}\mathcal{O}$ that contains 1 and is such that $\mathcal{Y}(A, 1) \cong \mathcal{O}$ (equivalently, $\{ y_1(a) \mid a \in A \}$ generates $\mathcal{O}$ as a frame).

(*b*) An **RL**-morphism $\beta : (\mathcal{E}, A) \to (\mathcal{F}, B)$ is a pair consisting of:

- (*i*) a frame morphism $\pi : \mathcal{E} \to \mathcal{F}$, and
- (*ii*) a proper unit $u \in \mathcal{R}\mathcal{F}$ such that $u \cdot (\pi \circ a) \in B$, for all $a \in A$.

Motivation. The name "**RL**" is intended to suggest the phrase"represented ℓ-group." The motivation here is create a means to record systematically all the data in all the possible morphisms $\Phi_e : A \longrightarrow \mathcal{R}\mathcal{Y}(a, e)$, as e varies over A^+.

Notation. $\mathcal{R}\mathcal{O}$ contains the constant function 1 as a distinguished weak unit. If a sub-ℓ-group $A \subseteq \mathcal{R}\mathcal{O}$ contains 1 and we want to draw attention to the fact that we are viewing 1 as an element of A, we write 1_A to denote it. If A is simply an abstract archimedean ℓ-group, then the notation 1_A is meaningless, but if A contains a weak unit e, then $1_{\Phi_e(A)} = \Phi_e(e) \in \Phi_e(A)$.

Definition. Suppose $1, e \in A^+ \subseteq \mathcal{R}\mathcal{O}$. We call e a *proper unit* if $y(e) = y(1)$.

Comment. Suppose $e, f \in A^+$. Even when both e and f are weak units (i.e., $e^\perp = \{0\} = f^\perp$), it may not be the case that $y(e) = y(f)$. In particular, when $A \subseteq \mathcal{R}\mathcal{O}$, there may be elements $a \in A$ such that $y(1) < y(a)$. We had an example of this previously: Represent $A = PL([0, 1])$ using x (the identity function from $[0, 1]$ to $\mathbb{R}$) as the weak unit. Then $\mathcal{Y}(A, x) = (0, 1]$. However, $1/x \in \mathcal{R}\mathcal{Y}(A, x)$ generates an archimedean kernel that is properly larger than $y(1)$. Observe that, $\Phi_x(x) = 1$, and $\Phi_x(1) = 1/x$.

Comment. Given $a : \mathcal{R} \longrightarrow \mathcal{E}$ with $a \in A$ and unit $u \in \mathcal{R}\mathcal{F}$, it is of course the case that $u \cdot (\pi \circ a) \in \mathcal{R}\mathcal{F}$. The definition of **RL** demands more: $u \cdot (\pi \circ a)$ *must be in* B. The data in the definition implies the existence of an ℓ-homomorphism $\rho : A \longrightarrow B$ defined by $\rho(a) = u \cdot (\pi \circ a)$. We may refer to an **RL**-morphism by the data (π, ρ), rather than (π, u). In general, $\rho(1_A)$ will not be equal to 1_B.

Research Problem. Does **RL** have limits (fiber products)? We can from limits of abelian ℓ-groups and limits of of regular Lindelöf locales; see Slide 10, below. But can we do so in a way that respects the rest of the structure in **RL**?

RL-presheaves

Let **C** be a category. An ***RL**-presheaf* on **C** is a functor $\mathbf{\Phi}$ from $\mathbf{C}^{op}$ to **RL**.

Notation. $\mathbf{\Phi}$ is the following data:

- For each $X \in \mathbf{C}$, $\mathbf{\Phi}(X) = (\mathcal{O}_X, A_X)$.
- For each $f : X \to Y \in \mathbf{C}$, an **RL**-morphism

$$\mathbf{\Phi}(f) = (\pi_f : \mathcal{O}_Y \to \mathcal{O}_X, \rho_f : A_Y \to A_X),$$

This assignment must of course preserve identity and composition.

Definition. Let $[A^+]$ denote the category whose objects are the elements of A^+, where the set $\hom(e, f)$ has a single element, denoted $f \preceq e$, if $y(f) \leqslant y(e)$ and is empty otherwise.

Fact. $\pi^e_e = \mathrm{id}_{\mathcal{Y}(A,e)}$ and $\rho^e_e = \mathrm{id}_{\Phi(A,e)}$.

Fact. Suppose $g \preceq f \preceq e$. Then $\pi^f_g \pi^e_f = \pi^e_g$ and $\rho^f_g \rho^e_f = \rho^e_g$.

Definition. Suppose A is an archimedean ℓ-group and E is a full subcategory of $[A^+]$. Then, the *representation presheaf for A over E* is the (contravariant) functor $\mathbf{\Phi}$ from E to $\mathbf{RL_1}$ defined (for $e, f \in E$) by:

(*i*) $\mathbf{\Phi}(e) := (\mathcal{Y}(A, e), \Phi(A, e))$, and

(*ii*) $\mathbf{\Phi}(f \preceq e) := (\pi^e_f, \rho^e_f)$.

As mentioned previously, we may regard $\Phi(a, e) \in \mathcal{RY}(a, e)$ as the "formal ratio of a to e." Then $\mathbf{\Phi}$ is an assemblage of data displaying all the formal ratios that can be formed with denominators in E and the relationships between them.

If it is necessary to keep track of the data defining $\mathbf{\Phi}$, we may write $\mathbf{\Phi}_E$, or for even more detail, $\mathbf{\Phi}_{(A,E)}$.

Natural Transformations of Representation Presheaves

Reminder: (i) $\mathbf{\Phi}_E(e) := (\mathcal{Y}(A,e), \Phi(A,e))$, and (ii) $\mathbf{\Phi}_E(f \preceq e) := (\pi^e_f, \rho^e_f)$.

Suppose $\beta : A \to B$ is an **Arch**-morphism and $E \subseteq [A^+]$. If $y(a) \leqslant y(a')$, then $y(\beta a) \leqslant y(\beta a')$. Thus, β is a functor from E to βE.

Note that $\mathbf{\Phi}_{(B,\beta E)}$ can be regarded as a composition of functors: $\mathbf{\Phi} \circ \beta$.

There is a natural transformation $\hat{\beta}$ from $\mathbf{\Phi}_{(A,E)}$ to $\mathbf{\Phi}_{(B,\beta E)}$ whose component at $e \in E$ is defined as follows:

$$\hat{\beta}_e := (\mathcal{Y}(\beta, e), \mathcal{Y}(\beta, e) \circ \_\,) : (\mathcal{Y}(A,e), \Phi(A,e)) \to (\mathcal{Y}(B,\beta e), \Phi(B,\beta e)).$$

$$\begin{array}{ccccc} e & & \mathbf{\Phi}_{(A,E)}(e) & \xrightarrow{\hat{\beta}_e} & \mathbf{\Phi}_{(B,\beta E)}(\beta e) \\ \preceq \uparrow & & \mathbf{\Phi}_{(A,E)}(f \preceq e) \downarrow & & \downarrow \mathbf{\Phi}_{(B,\beta E)}(\beta f \preceq \beta e) \\ f & & \mathbf{\Phi}_{(A,E)}(f) & \xrightarrow{\hat{\beta}_f} & \mathbf{\Phi}_{(B,\beta E)}(\beta f) \end{array}$$

Recovering A

We return to the Research Question form Slide 6.

Consider a presheaf $\mathbf{\Phi}_{(A,E)}$.

- The maps $\rho^e_f : \Phi(A,e) \to \Phi(A,f)$ for all $e,f \in E$, $f \preceq e$ form a diagram in **Arch**. For each $e \in E$, there is a surjective **Arch**-morphism $\Phi_e : A \to \Phi(A,e)$. If E is cofinal in A^+, then for any $a \in A$, there is $e \in E$ such that $y(a) \leqslant y(e)$. It follows that A, together with the maps Φ_e, form the limit of the ρ-diagram.
- Similarly, the maps $\pi^e_f : \mathcal{Y}(A,e) \to \mathcal{Y}(A,f)$ for all $e,f \in E$, $f \preceq e$ form a diagram in **RegLin**. We conjecture that this too has a limit. We do not know if $\mathcal{Y}A$ is regular, but if it is, then the limit would be the Lindelöfification $\lambda\mathcal{Y}A$ of $\mathcal{Y}A$.
- **Question.** Is there (always) an embedding $\Phi : A \to \mathcal{R}\lambda\mathcal{Y}A$ and a collection of "scaled restriction maps" $\rho_e : \Phi A \to \Phi(A,e)$?

An Example of Conrad-Martinez (simplified by Hager-Johnson)

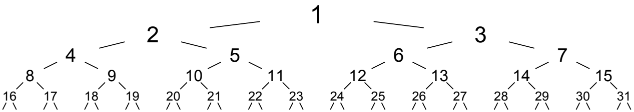
Let $\mathcal{M}$ be a family of infinite subsets of $\mathbb{N}$. For each $M \in \mathcal{M}$, let $\gamma_M \in \mathbb{R}^{\mathbb{N}}$. Then, $G(\mathcal{M}, \gamma)$ denotes the sub-ℓ-group of $\mathbb{R}^{\mathbb{N}}$ generated by $\{ \chi_M \cdot \gamma_M \mid M \in \mathcal{M} \} \cup \{ \chi_{\{n\}} \mid n \in \mathbb{N} \}$.

Lemma. Suppose Γ is cofinal in $\mathrm{Inc}(\mathbb{N}, \mathbb{N})$, the set of strictly-increasing sequences. For any $u \in (\mathbb{R}_{>0})^{\mathbb{N}}$, there is $\gamma \in \Gamma$ such that $u \cdot \gamma$ is unbounded.

Proof. Given u, pick $w \in \mathbb{R}^{\mathbb{N}}$ such that $w \leqslant u$ and $1/w \in \mathrm{Inc}(\mathbb{N}, \mathbb{N})$. Pick $\gamma \in \Gamma$ such that $\gamma \geqslant (1/w)^2$. Then, $u \cdot \gamma \geqslant u \cdot (1/w)^2 \geqslant 1/w$. □

Corollary. Suppose $\{ \gamma_M \mid M \in \mathcal{M} \}$ is contained in and cofinal in $\mathrm{Inc}(\mathbb{N}, \mathbb{N})$. For any $u \in (\mathbb{R}_{>0})^{\mathbb{N}}$, $u\, G(\mathcal{M}, \gamma)$ contains an unbounded sequence.

Fact. There is $\mathcal{M}$ such that $M_0 \cap M_1$ finite for all distinct $M_0, M_1 \in \mathcal{M}$, and $|\mathcal{M}| = \mathfrak{c}$. For example, each of the $\mathfrak{c}$ branches in the infinite binary tree whose first few levels are shown below contains an infinite subset of $\mathbb{N}$, and any two branches have finite intersection:



For $\mathcal{M}$ as in the Fact, it can be shown that $G(\mathcal{M}, \gamma)$ is hyperarchimedean. If in addition, γ is as in the corollary, then $G(\mathcal{M}, \gamma)$ is not contained in a unital hyperarchimedean ℓ-group. Thus, there is a hyperarchimedean ℓ-group without unit that cannot be embedded in an hyperarchimedean ℓ-group with unit (as Conrad and Martinez showed).

References: "Conrad-Martinez-1990.pdf", "Hager-Johnson-2010.pdf"

Notes on previous slide

Order $\mathbb{R}^{\mathbb{N}}$ pointwise.

Fact. $\mathrm{Inc}(\mathbb{N}, \mathbb{N})$ is cofinal in $\mathbb{R}^{\mathbb{N}}$.
Proof. For $f \in \mathbb{R}^{\mathbb{N}}$, define $[\![f]\!] \in \mathbb{R}^{\mathbb{N}}$ by $[\![f]\!](n) := \bigvee_{i=0}^{n} \lceil f(i) \rceil$.

Fact. Suppose $u \in (\mathbb{R}_{>0})^{\mathbb{N}}$. If C is cofinal in $\mathbb{R}^{\mathbb{N}}$, then so is uC.
Proof. Let $g \in \mathbb{R}^{\mathbb{N}}$. Pick $c \in C$ such that $g/u \leqslant c$. Then $g \leqslant uc$.

Representing the Conrad-Martinez-Hager-Johnson ℓ-groups

Suppose

- $\mathcal{M} \subseteq \mathcal{P}\mathbb{N}$ such that $L \cap M$ is finite for all distinct $L, M \in \mathcal{M}$, and
- $\gamma : M \mapsto \gamma_M : M \to \mathrm{Inc}(\mathbb{N}, \mathbb{N})$ has cofinal image.

Let $G := G(\mathcal{M}, \gamma)$ be the sub-ℓ-group of $\mathbb{R}^{\mathbb{N}}$ generated by

$$\{ g_M := \chi_M \cdot \gamma_M \mid M \in \mathcal{M} \} \cup \{ \chi_{\{n\}} \mid n \in \mathbb{N} \}.$$

Lemma. Each element $g \in G$ can be expressed in the form $g = h + \sum_{B \in \mathcal{B}} b_B g_B$, where h has finite support, $\mathcal{B} \subseteq \mathcal{M}$ is finite, and $b_B \in \mathbb{Z} \setminus \{0\}$. If $h + \sum_{B \in \mathcal{B}} b_B g_B = h' + \sum_{B \in \mathcal{B}'} b'_B g_B$, then $\mathcal{B} = \mathcal{B}'$ and $b_B = b'_B$ for all $B \in \mathcal{B}$. □

Suppose $\mathcal{B} \subseteq \mathcal{M}$ is finite. Let $N(\mathcal{B}) := \max \bigcup \{ L \cap M \mid L \neq M,\ L, M \in \mathcal{B} \}$. Define $g_{\mathcal{B}}$ by

$$g_{\mathcal{B}}(i) = \begin{cases} 1, & \text{if } i \leqslant N(\mathcal{B}); \\ \gamma_M(i), & \text{if } i > N(\mathcal{B}) \text{ and } i \in M. \end{cases}$$

Fact. $\{ y(g_{\mathcal{B}}) \mid \mathcal{B} \text{ a finite subset of } \mathcal{M} \}$ is cofinal in $\mathcal{Y}G$.

Fact. $\mathcal{Y}(G, g_{\mathcal{B}})$ is homeomorphic the disjoint union of $\{ n \in \mathbb{N} \mid n \leqslant N(\mathcal{B}) \}$ and the sets $\overline{B}$, $B \in \mathcal{B}$, where

$$\overline{B} := \{ b \in B \mid b > N(\mathcal{B}) \} \cup \{ p_B \}$$

is the one-point compactification of $\{ b \in B \mid b > N(\mathcal{B}) \}$.

Fact. For $i \in [0, n(\mathcal{B})] \cup \bigcup \mathcal{B}$,

$$\Phi(g, g_{\mathcal{B}})(i) = \begin{cases} g(i), & \text{if } i \leqslant N(\mathcal{B}); \\ g(i)/\gamma_B(i), & \text{if } i > N(\mathcal{B}) \text{ and } i \in B; \end{cases}$$