A Course on the Yosida Theorem
Classical & Pointfree Versions & Applications

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Lecture 4. Applications: 
Archimedean $\ell$-Groups with Strong Unit 
Monoreflections

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The Yosida Representation (review of last lecture)

An important and influential reference for the material in Lectures 2 & 3 (and more) is:


A copy is in the Dropbox folder ("Hager-Robertson-1975.pdf")

In the present lecture, we will review the Yosida Representation, and then apply it to the category $W^*$ of archimedean $\ell$-groups with strong unit. A major goal of much research over the past several decades has been to generalize results concerning $W^*$ to the categories $W$ (archimedean $\ell$-groups with weak unit) and $Arch$ (archimedean $\ell$-groups in general). In this lecture, we will present some of the $W^*$ prototype theorems.
Suppose $A$ is an archimedean $\ell$-group with weak unit $u$. Then there is a compact Hausdorff space $Y(A, u)$ and an $\ell$-isomorphism:

$$\hat{\ )}_u : A \rightarrow \hat{A}_u \subseteq D(Y(A, u)),$$

where $\hat{A}_u$ is a point-separating* $\ell$-group of continuous extended-real-valued functions on $Y(A, u)$ and $\hat{u}_u = 1$.

* We say a set $A$ of functions in $D(X)$ is point-separating if for any $x_0, x_1 \in X$, there is $a \in A$ such that $a(x_0) = 0$ and $a(x_1) \neq 0$. 
Suppose $B$ is an archimedean $\ell$-group with weak unit $v$, and $\phi : A \to B$ is an $\ell$-group morphism with $\phi(u) = v$.

Then there is a continuous map: $Y(\phi) : Y(B, v) \to Y(A, u)$ such that

$$\hat{\phi}(a)_v = \hat{a}_u \circ Y(\phi).$$
Suppose $A$ is a point-separating $\ell$-group of continuous extended-real-valued functions on a compact Hausdorff space $X$, and $1 \in A$.

Then $x \mapsto M_x := \{ a \in A \mid a(x) = 0 \}$ is a homeomorphism of $X$ with $Y(A, 1)$, and $\widehat{a}_1(M_x) = a(x)$ for all $a \in A$. 

The Categories $\mathbf{W}$ and $\widehat{\mathbf{W}}$

Objects of $\mathbf{W}$: Pairs $(A, u)$, $A$ an archimedean $\ell$-group; $u \in A$ a weak unit.

Morphisms of $\mathbf{W}$: $\ell$-group morphisms $\phi : (A, u) \rightarrow (A', u')$, with $\phi(u) = u'$.

Objects of $\widehat{\mathbf{W}}$: Pairs $(A, X)$, where $X$ is a compact Hausdorff space and $A$ is a point-separating $\ell$-group of continuous extended-real-valued functions on $X$ containing 1.

Morphisms of $\widehat{\mathbf{W}}$: A morphism from $(A, X)$ to $(A', X')$ is a continuous map $\Phi : X' \rightarrow X$ such that for all $a \in A$, $a \circ \Phi \in A'$.

This implies in particular that $\Phi^{-1}(a^{-1}(+\infty))$ must be nowhere dense in $X'$ for all $a \in A$; see (Hager Robertson 1975, Remark 2.13). $A$, as an algebra of functions on $X$, endows $X$ with a geometric structure. The condition $a \circ \Phi \in A'$ is a demand that $\Phi$ preserve this structure. This is an embryonic version of idea underlying the idea of a “structure sheaf” — a topic that we will explore in detail later.
Yosida Theorem: Categorical Interpretation

The Yosida Theorem says that the categories $\mathcal{W}$ and $\widehat{\mathcal{W}}$ are equivalent, via the functor $Y$, where

$$Y(A, u) := (\widehat{A}_u, Y(A, u))$$
$$Y(\phi) := Y(\phi)$$

Because of this, we need not distinguish between the two categories. Given any $\phi : (A, u) \to (B, v)$ in $\mathcal{W}$, we may assume without any loss of generality that

- $A$ and $B$ are $\ell$-groups of continuous extended-real-valued functions on spaces $Y(A, u)$ and $Y(B, v)$ and
- $\phi$ is induced by a continuous map $f : Y(B, v) \to Y(A, u)$. 

Definition. Let $A$ be an $\ell$-group. An element $e \in A$ is called a strong unit if $0 \leq e$ and for every $a \in A$, there is $n \in \mathbb{N}$ such that $|a| \leq n e$.

Definition.

(i) The category $\mathcal{W}^*$ is the full subcategory of $\mathcal{W}$ whose objects are those pairs $(A, e)$ such that $e$ is a strong unit.

(ii) The category $\mathcal{C}^*$ is the full subcategory of $\mathcal{W}^*$ whose objects are those pairs $(A, e)$ such that $\hat{A}_e = \mathcal{C}^*(Y(A, e))$ (all continuous real-valued functions on $Y(A, e)$—necessarily bounded, since $Y(A, e)$ is compact).
**C** is a monoreflective subcategory of **W**

Suppose \((A, e)\) is an object of **W**. Then we have a **W** morphism

\[ \rho_A : A \to \rho A := C^*(Y(A, e)), \text{ where } \rho_A(a) := \hat{a}_e. \]

**Comment:** We have merely renamed a few things that we’ve already encountered — \(\rho_A\) is the representation morphism \((\hat{\cdot})_e\). We do this to be able say things in a manner that meshes with category theory. In particular, we will show that \(\rho_A\) is a **reflection map**—a concept we will define on the next slide.

**Theorem.** Let \((B, d)\) be an object of **C***, and let \(\phi : (A, e) \to (B, d)\) be a morphism of **W**. Then there is a **C***-morphism \(\overline{\phi} : (\rho A, e) \to (B, d)\) such that \(\phi = \overline{\phi} \circ \rho_A\).

**Proof.** Without loss of generality, we may assume \(B = C^*(Y(B, d))\). \(\phi\) is induced by a continuous map \(f : Y(B, d) \to Y(A, e)\), such that \(\overline{\phi(a)}_d = \hat{a}_e \circ f\). Let \(c\) be any continuous function on \(Y(A, e)\). Since \(\hat{c}_e \circ f\) is a continuous, real-valued function on \(Y(B, d)\), we have \(\hat{c}_e \circ f \in B\). Thus, we may define \(\overline{\phi}\) by \(\overline{\phi}(c) = \hat{c}_e \circ f\), for \(c \in \rho A\).
Reflective subcategories

Definition. Suppose $\mathbf{C}$ is a category. We say $\mathbf{R}$ is reflective in $\mathbf{C}$ if:

- $\mathbf{R}$ is a full, isomorphism- closed subcategory of $\mathbf{C}$, and
- for each object $A$ of $\mathbf{C}$, there is a morphism $\rho_A : A \to \rho A$ that is universal to $\mathbf{R}$, i.e., $\rho A$ is in $\mathbf{R}$ and for any $\mathbf{C}$-morphism $\phi : A \to B$ with codomain $B$ in $\mathbf{R}$, there is a unique morphism $\overline{\phi} : \rho A \to B$, with $\phi = \overline{\phi} \circ \rho A$.

If $\rho_A$ is monic (i.e., left-cancellable) for all $A$, then $\mathbf{R}$ is said to be monoreflective. Some examples of monoreflective subcategories:

<table>
<thead>
<tr>
<th>$\mathbf{C}$</th>
<th>Tych. sp.</th>
<th>distr. latt.</th>
<th>torsion-free ab. grps</th>
<th>$\mathbf{W}^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbf{R}$</td>
<td>cpt $T_2$ sp.</td>
<td>bool. latt.</td>
<td>divisible t-f ab. grps</td>
<td>$\mathbf{C}^*$</td>
</tr>
</tbody>
</table>
Free objects in $W^*$

**Definition.** Let $I := [0, 1] \subseteq \mathbb{R}$. Let $S$ be a set. For $\alpha \in S$, let $x_\alpha : I^S \to I$ be the projection onto the $\alpha^{th}$ factor, and let $F_\ell^*(S)$ denote the sub-$\ell$-group of $C(I^S)$ generated by $\{x_\alpha \mid \alpha \in S\} \cup \{1\}$. Each element of $F_\ell^*(S)$ is of the form $\bigvee_{i=1}^m \bigwedge_{j=1}^{n_i} f_{ij}$, where $m, n_i \in \mathbb{N}$ and each $f_{ij}$ is an integer plus an integer-linear combination of finitely many elements of $\{x_\alpha \mid \alpha \in S\} \cup \{1\}$.

**Proposition.** Suppose $(A, e) \in W^*$ and $\alpha \mapsto a_\alpha$ is a function from $S$ to $A$, with $0 \leq a_\alpha \leq e$ for all $\alpha \in E$. Then, there is a unique $\ell$-group morphism $\phi : F_\ell^*(S) \to A$ such that $\phi(1) = e$ and $\phi(x_\alpha) = a_\alpha$ for all $\alpha \in S$.

**Proof.** For each $\alpha \in S$, $\widehat{a_\alpha e} : Y(A, e) \to I$. Consider the map $\Phi : Y(A, e) \to I^S$ whose component at $\alpha$ is $\widehat{a_\alpha e}$. For all $f \in F_\ell^*(S)$ define $\phi(f) := f \circ \Phi$. Clearly, $x_\alpha \circ \Phi = \widehat{a_\alpha e}$. Moreover, if $f, g \in F_\ell^*(S)$, then $(f \lor g) \circ \Phi = (f \circ \Phi) \lor (g \circ \Phi)$ and $(f + g) \circ \Phi = (f \circ \Phi) + (g \circ \Phi)$, so $\phi$ is an $\ell$-group morphism. \qed
Epimorphisms in $W^*$

**Definition.** A morphism $\epsilon : A \rightarrow B$ in a category $C$ is said to be $C$-epi if for all $C$-morphisms $\phi, \psi : B \rightarrow C$, $\phi \circ \epsilon = \psi \circ \epsilon$ implies $\phi = \psi$.

**Examples.** Set-epi = surjective. A morphism $f : A \rightarrow B$ of bounded distributive lattices is epi if every element of $B$ is either in $f(A)$ or is the complement of an element of $f(A)$.

**Proposition.** A $W^*$-morphism $\epsilon : (A, a) \rightarrow (B, b)$ is epi iff $Y(\epsilon) : Y(B, b) \rightarrow Y(A, a)$ is injective.

**Proof Sketch.** $\epsilon$ is $W^*$-epi iff $\rho \epsilon$ is $C^*$-epi iff $Y(\rho \epsilon)$ is $\text{CptHaus}$-mono iff $Y(\rho \epsilon) = Y(\epsilon)$ is injective.

**Cor.** An object $(A, a)$ of $W^*$ is epicomplete iff it is a $C^*(X)$ for some compact Hausdorff space $X$. 
Some guiding problems suggested by $W^*$

In the lectures that follow, our goal will be to develop the theory needed to address the following:

▶ Is there a functorial representation for $\text{Arch}$? (Unsolved.)

▶ What are the free objects in $W$? In $\text{Arch}$? (Solved, and easy to represent.)

▶ What are the epimorphisms in $W$? In $\text{Arch}$? (Solved by Ball & Hager. Not easy.)

▶ What are the epi-closed objects in $W$? In $\text{Arch}$? (Solved for $W$ by Madden & Vermeer using locales. Solved for $W$ and $\text{Arch}$ by Ball & Hager without locales. Not easy.)

▶ What can we say about the monoreflective subcategories of $W$? (Much is known.) Of $\text{Arch}$? (Important problems are unsolved.)