ORTHOGONAL FUNCTIONS: THE LEGENDRE, LAGUERRE, AND HERMITE POLYNOMIALS

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Abstract. The Legendre, Laguerre, and Hermite equations are all homogeneous second order Sturm-Liouville equations. Using the Sturm-Liouville Theory we will be able to show that polynomial solutions to these equations are orthogonal. In a more general context, finding that these solutions are orthogonal allows us to write a function as a Fourier series with respect to these solutions.

1. Introduction

The Legendre, Laguerre, and Hermite equations have many real world practical uses which we will not discuss here. We will only focus on the methods of solution and use in a mathematical sense. In solving these equations explicit solutions cannot be found. That is solutions in in terms of elementary functions cannot be found. In many cases it is easier to find a numerical or series solution.

There is a generalized Fourier series theory which allows one to write a function $f(x)$ as a linear combination of an orthogonal system of functions $\phi_1(x)$,$\phi_2(x)$,...,$\phi_n(x)$,... on $[a, b]$. The series produced is called the Fourier series with respect to the orthogonal system. While the coefficients ,which can be determined by the formula $c_n = \frac{\int_a^b f(x)\phi_n(x)dx}{\int_a^b \phi_n^2(x)dx}$, are called the Fourier coefficients with respect to the orthogonal system. We are concerned only with showing that the Legendre, Laguerre, and Hermite polynomial solutions are orthogonal and can thus be used to form a Fourier series. In order to proceed we must define an inner product and define what it means for a linear operator to be self-adjoint.

Definition 1. *We define an inner product* $\langle y_1|y_2\rangle = \int_a^b y_1(x)\overline{y_2(x)}dx$ *where* $y_1, y_2 \in C^2[a, b]$ *and two functions aresaid to be orthogonal if* $(y_1|y_2) = \int_a^b y_1(x)\overline{y_2(x)} = 0$.

Definition 2. *A linear operator* L *is self-adjoint if* $\langle Ly_1|y_2\rangle = \langle y_1|Ly_2\rangle$ *for all* y_1,y_2.

2. The Sturm-Liouville Theory

A Sturm-Liouville equation is a homogeneous second order differential equation of the form

$$[p(x)y']' + q(x)y + \lambda r(x)y = 0 \tag{2.1}$$

where $p(x)$, $r(x) > 0$ on the interval $[a, b]$ and where the function $q(x)$ is real-valued. In order to make the problem simpler to solve we assume $p(x)$, $p'(x)$, $r(x)$, $q(x) \in C[a, b]$. We rewrite the equation in the form of an eigenvalue equation by defining a linear operator L on $C^2[a, b]$ as

$$Ly = [p(x)y']' + q(x)y. \tag{2.2}$$

Once Ly is defined, the Sturm-Liouville equation can be written in the form

$$Ly + \lambda r(x)y = 0 \tag{2.3}$$

Now we impose boundary conditions such that $y \in C^2[a, b]$ so that L will be self-adjoint with respect to the inner product defined above which allows us to rewrite differential equations of the same form to show that its solutions $y_1, y_2 \in C^2[a, b]$ form an orthogonal basis. It is also necessary to note that if $y \neq 0$ and $y \in BC^2[a, b]$ is a solution to $Ly + \lambda ry = 0$ then y is an eigenfunction and λ is an eigenvalue. Therefore (y, λ) is an eigenpair.

Remark. We want to know the boundary conditions necessary for L to be self adjoint. We want $\langle Ly_1|y_2\rangle - \langle y_1|Ly_2\rangle = 0$. Note that

$$\begin{aligned}
&\langle [p(x)y_1']' + q(x)y_1|y_2\rangle - \langle y_1|[p(x)y_2']' + q(x)y_2\rangle \\
&= \int_a^b (p'y_1'\overline{y_2} + py_1''\overline{y_2} + qy_1\overline{y_2} - y_1p'\overline{y_2'} - y_1p\overline{y_2''} - y_1\overline{q_1y_2})dx \\
&= \int_a^b (p'y_1'\overline{y_2} + py_1''\overline{y_2} - y_1p'\overline{y_2'} - y_1p\overline{y_2''})dx \\
&= \int_a^b [p(y_1'\overline{y_2} - \overline{y_2'}y_1)]'dx \\
&= p(b)(y_1'(b)\overline{y_2}(b) - \overline{y_2'}(b)y_1(b)) - p(a)(y_1(a)\overline{y_2}(a) - \overline{y_2'}(a)y_1(a))
\end{aligned}$$

In order for the equality to hold we wish to impose the boundary conditions $y(a) = y(b) = 0$ $y'(a) = y'(b) = 0$. With these conditions we say that $y \in BC^2[a, b]$.

Lemma 2.1. *The eigenvalues of a Sturm-Liouville problem are real.*

Proof. Suppose $y \neq 0$, $y \in BC^2[a,b]$ is a solution and satisfies $Ly + \lambda r y = 0$ and compute $\langle Ly|y\rangle = \langle y|Ly\rangle$.

$$\begin{aligned}
Ly &= -\lambda r y \\
\langle -\lambda r y|y\rangle &= \langle y| -\lambda r y\rangle \\
-\langle \lambda r y|y\rangle &= -\langle y|\lambda r y\rangle \\
\langle \lambda r y|y\rangle &= \langle y|\lambda r y\rangle \\
\lambda\langle r y|y\rangle &= \overline{\lambda}\langle y|r y\rangle \\
\lambda \int_a^b y\overline{y} r(x)dx &= \overline{\lambda} \int_a^b y\overline{y} r(x)dx \\
\lambda \int_a^b |y(x)|^2 r(x)dx &= \overline{\lambda} \int_a^b |y(x)|^2 r(x)dx \\
\overline{\lambda} &= \lambda \\
\therefore \lambda &\in \mathbb{R}
\end{aligned}$$

□

With this equality, we have a new inner product called the weighted inner product

$$\langle y_1|y_2\rangle_r = \int_a^b y_1(x)\overline{y_2(x)} r(x)dx \tag{2.4}$$

where $y_1, y_2 \in C^2[a,b]$ and $(y|y) > 0$ when $y \neq 0$.

Lemma 2.2. *If (y_1,λ_1), (y_2,λ_2) are eigenpairs where $\lambda_1 \neq \lambda_2$ then y_1 and y_2 are orthogonal.*

Proof. We know that L is self-adjoint becasue $y \in BC^2[a,b]$, $Ly = -\lambda r y$, and $\lambda \in \mathbb{R}$.

$$\begin{aligned}
\langle Ly_1|y_2\rangle &= \langle y_1|Ly_2\rangle \\
\langle -\lambda_1 r y_1|y_2\rangle &= \langle y_1| - \lambda_2 r y_2\rangle \\
-\langle \lambda_1 r y_1|y_2\rangle &= -\langle y_1|\lambda_2 r y_2\rangle \\
\langle \lambda_1 r y_1|y_2\rangle &= \langle y_1|\lambda_2 r y_2\rangle \\
\lambda_1\langle ry|y\rangle &= \lambda_2\langle y_1|ry_2\rangle \\
\lambda_1\int_a^b y_1\overline{y_2}r(x)dx &= \lambda_2\int_a^b y_1\overline{y_2}r(x)dx \\
\lambda_1\langle y_1|y_2\rangle_r &= \lambda_2\langle y_1|y_2\rangle_r \\
(\lambda_1-\lambda_2)\langle y_1|y_2\rangle_r &= 0 \\
\langle y_1, y_2\rangle_r &= 0
\end{aligned}$$

□

Therefore y_1 and y_2 are orthogonal.

3. The Legendre Polynomials

The Legendre Differential Equation is

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0, n \in \mathbb{R}, x \in (-1,1) \tag{3.1}$$

We know that $x = 0$ is an ordinary point of equation (3.1). We see that when we divide by the coefficient $(1-x^2)$ that $x \in (-1,1)$. We will see later that the property of orthogonality falls out on the interval $[-1,1]$ by the Sturm-Liouville Theory. In order to find the series solution to this differential equation we will use the power series method.

$$\begin{aligned}
\text{Let } y(x) &= \sum_{k=0}^{\infty} a_k x^k \\
y'(x) &= \sum_{k=1}^{\infty} a_k k x^{k-1} \\
y''(x) &= \sum_{k=2}^{\infty} a_k k(k-1)x^{k-2}
\end{aligned}$$

Insert these terms into the original equation (3.1) to obtain

$$(1-x^2)\sum_{k=2}^{\infty} a_k k(k-1)x^{k-2} - 2x\sum_{k=1}^{\infty} a_k k x^{k-1} + n(n+1)\sum_{k=0}^{\infty} a_k x^k = 0.$$

which gives

$$\sum_{k=2}^{\infty} a_k k(k-1)x^{k-2} - \sum_{k=2}^{\infty} a_k k(k-1)x^k - 2\sum_{k=1}^{\infty} a_k k x^k + n(n+1)\sum_{k=0}^{\infty} a_k x^k = 0$$

making powers and indicies equal

$$\sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k - \sum_{k=0}^{\infty} a_k k(k-1)x^k - 2\sum_{k=0}^{\infty} a_k(k)x^k + \sum_{k=0}^{\infty} n(n+1)a_k x^k = 0$$

simplify

$$\sum_{k=0}^{\infty}[(k+2)(k+1)a_{k+2} - (k)(k-1)a_k - 2ka_k + n(n+1)a_k]x^k = 0$$

equating coefficients

$$(k+2)(k+1)a_{k+2} - (k)(k-1)a_k - 2ka_k + n(n+1)a_k = 0$$

solving for a_{k+2} gives us a recurrence relation

$$a_{k+2} = \frac{k(k+1)-n(n+1)}{(k+2)(k+1)}a_k. \tag{3.2}$$

Remark. We are looking for polynomial solutions. If we assume our solution has degree L then

$$\sum_{k=0}^{L} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots + a_L + 0x^{L+1} + 0x^{L+2} + \cdots$$

Where all the terms following a_L will be zero, while $a_L \neq 0$. So we know,

$$\begin{aligned} a_{L+2} = a_L \frac{L(L+1) - n(n-1)}{(L+2)(L+1)} &= 0 \\ \frac{L(L+1) - n(n-1)}{(L+2)(L+1)} &= 0 \\ L(L+1) - n(n+1) &= 0 \\ L(L+1) &= n(n+1) \\ L = n \text{ or } L &= -(n+1) \end{aligned}$$

So $L = n$ is our solution because all terms after $n+1$ are zero. Therefore the degree of our polynomial solution is n where n is an integer.

We get two linearly independent series solutions from the recurrence relation (3.2). The first solutions comes from the even values of k. While the second solution comes from the odds values of k. We assume $a_0 \neq 0$ and $a_1 \neq 0$.

$$\begin{aligned} y_1(x) = a_0[1 &- \frac{n(n+1)}{2}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 \\ &- \frac{(n-4)(n-2)n(n+1)(n+3)(n+5)}{6!}x^6 + \cdots] \end{aligned}$$

$$y_2(x) = a_1[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 + \cdots]$$

Where both solutions are valid for $x \in (-1, 1)$. Finding the Legendre polynomials can be very long and difficult. There are many methods including Rodrigue's Formula that are useful in finding these polynomials. The first five Legendre Polynomials turn out to be

$$\begin{aligned}
P_0(x) &= 1\\
P_1(x) &= x\\
P_2(x) &= \frac{1}{2}(3x^2-1)\\
P_3(x) &= \frac{1}{2}x(5x^2-3)\\
P_4(x) &= \frac{1}{8}(35x^4-30x^2+3)\\
&.\\
&.\\
&.
\end{aligned}$$

By rewriting the Legendre Polynomial as a Sturm-Liouville problem, we can prove its orthgonality. We find that the operator can be written as

$$Ly = [(1-x^2)y']'.$$

where $f, g \in C[-1,1]$. After imposing the conditions that any $f, g \in BC^2[-1,1]$ whenever f, g meet the conditions. We want $\langle Lf|g\rangle = \langle f|Lg\rangle$. That is we want L to be self-adjoint.

$$\begin{aligned}
&\langle Lf|g\rangle - \langle f|Lg\rangle = 0\\
&\int_{-1}^{1} Lf(x)g(x) - f(x)Lg(x)dx\\
&= \int_{-1}^{1} ((1-x^2)f')'g(x) - f(x)((1-x^2)g')'dx\\
&= \int_{-1}^{1} (-2xf' + (1-x^2)f'')g - f(-2xg' - (1-x^2)g'')dx\\
&= \int_{-1}^{1} (1-x^2)f''g - 2xf'g + 2xfg' - (1-x^2)fg''dx\\
&= \int_{-1}^{1} [(1-x^2)(f'g-g'f)]'dx\\
&= [(1-x^2)(f'g-g'f)]_{-1}^{1}\\
&= 0.
\end{aligned}$$

Therefore L is self-adjoint with no imposed conditions.

Let y_n and y_m ,where $n \neq m$, be polynomial solutions to the differential equation, $Ly_n = -n(n+1)y$.

$$\begin{aligned} & -n(n+1)\langle y_n|y_m\rangle \\ &= \langle Ly_n|y_m\rangle \\ &= \langle y_n|Ly_m\rangle \\ &= \langle y_n|-m(m+1)y_m\rangle \\ &= -m(m+1)\langle y_n|y_m\rangle \end{aligned}$$

So, $-n(n+1)\langle y_n|y_m\rangle = -m(m+1)\langle y_n|y_m\rangle$, since $n \neq m$, $\langle y_n|y_m\rangle = 0$. We could have also used Lemma (2.2) to say that the Legendre polynomials are orthogonal due to the Sturm-Liouville theory. The Legendre polynomaials are orthogonal on the interval $[-1,1]$ with respect to the the weight function $r(x) = 1$.

4. The Laguerre Polynomials

The Laguerre differential equation is

$$xy'' + (1-x)y' + ny = 0,\ n \in \mathbb{R},\ x \in [0,\infty). \tag{4.1}$$

We know that $x = 0$ is a regular singular point of equation (4.1). In order to find the series solution to this differential equation we must use the Frobenius method which is useful for solving equations of the form

$$x^2y'' + xp(x)y' + q(x)y = 0. \tag{4.2}$$

We will use the Frobenius method to find a series solution to equation (4.1) of the form

$$y_1(x) = x^r \sum_{k=0}^{\infty} a_k x^k,\ a_0 \neq 0 \tag{4.3}$$

where r is the root of the indicial equation

$$r(r-1) + p_0 r + q_0. \tag{4.4}$$

We compare the Laguerre equation(4.1) to our standard form equation (4.2). We multiply (4.1) by x to obtain the equation

$$x^2 + (1-x)xy' + nxy \tag{4.5}$$

We see that

$$\begin{aligned} p(x) = (1-x) \quad &\text{and} \quad q(x) = nx \\ p(0) = 1 \quad &\text{and} \quad q(0) = 0. \end{aligned}$$

Hence our indicial equation is

$$r(r-1) + r = 0$$

or

$$r^2 = 0.$$

Thus the roots of the indcial equation are $r_1 = r_2 = 0$. Both roots are equal so we will have a second linearly independent equation, which we will not use, of the the form

$$y_2(x) = y_1(x)\ln x + x^{r_1} \sum_{k=0}^{\infty} b_k x^k. \tag{4.6}$$

We will now find the recurrence realtion for the coefficients for $y_1(x)$ by direct substitution of $y_1(x)$ into equation (4.1).

$$\begin{aligned} y(x) &= x^r \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k x^{(k+r)} \\ y'(x) &= \sum_{k=1}^{\infty} (k+r) a_k x^{(k+r-1)} \\ y''(x) &= \sum_{k=2}^{\infty} (k+r)(k+r-1) a_k x^{(k+r-2)} \end{aligned}$$

Plug above equations into (4.1) to get

$$x\sum_{k=2}^{\infty}(k+r)(k+r-1)a_kx^{(k+r-2)}+(1-x)\sum_{k=1}^{\infty}(k+r)a_kx^{(k+r-1)} + n\sum_{k=0}^{\infty}a_kx^{(k+r)}=0$$

which gives us

$$\sum_{k=2}^{\infty}(k+r)(k+r-1)a_kx^{(k+r-1)}+\sum_{k=1}^{\infty}(k+r)a_kx^{(k+r-1)} - \sum_{k=1}^{\infty}(k+r)a_kx^{(k+r)} + \sum_{k=0}^{\infty}na_kx^{(k+r)}=0$$

making all powers equal

$$\sum_{k=1}^{\infty}(k+r+1)(k+r)a_{k+1}x^{(k+r)}+\sum_{k=0}^{\infty}(k+r+1)a_{k+1}x^{(k+r)} - \sum_{k=1}^{\infty}(k+r)a_kx^{(k+r)} + \sum_{k=0}^{\infty}na_kx^{(k+r)}=0$$

set r=0 to get

$$
\begin{aligned}
\sum_{k=1}^{\infty}(k+1)(k)a_{k+1}x^k + &\sum_{k=0}^{\infty}(k+1)a_{k+1}x^k \\
-&\sum_{k=1}^{\infty} ka_k x^k \\
+&\sum_{k=0}^{\infty} na_k x^k = 0.
\end{aligned}
$$

for the terms with $k = 1$ we see that the $k = 0$ terms add nothing so

$$
\sum_{k=0}^{\infty}[(k+1)(k)a_{k+1} + (k+1)a_{k+1} - ka_k + na_k]x^k = 0.
$$

Due to the uniqueness of a power series we set the coefficients equal to zero and solve for the recurrence relation.

$$
\begin{aligned}
&[(k+1)(k)a_{k+1} + (k+1)a_{k+1} - ka_k + na_k] = 0 \\
&[(k+1)(k) + (k+1)]a_{k+1} + (-k+n)a_k = 0 \\
&a_{k+1} = \frac{(k-n)a_k}{[(k+1)(k) + (k+1)]}
\end{aligned}
$$

The coefficients are

$$
\begin{aligned}
a_1 &= -na_0 \\
a_2 &= \frac{(1-n)a_1}{4} = \frac{n(n-1)a_0}{2^2} \\
a_3 &= \frac{(2-n)a_2}{9} = \frac{-n(n-1)(n-2)a_0}{2^2 \cdot 3^2} \\
a_4 &= \frac{(3-n)a_3}{16} = \frac{n(n-1)(n-2)(n-3)a_0}{2^2 \cdot 3^2 \cdot 4^2} \\
&. \\
&. \\
&. \\
a_k &= \frac{[(k-1)-n]a_{k-1}}{k^2} = \frac{(-1)^k n(n-1)(n-2)...(n-k+1)}{2^2 \cdot 3^2 \cdot 4^2 \cdots k^2} \\
&= \frac{(-1)^k n! a_0}{(k!)^2 (n-k)!}
\end{aligned}
$$

Therefore by substituting these coefficients into equation (4.3) we obtain the series solution for equation (4.1).

$$
\begin{aligned}
y_n(x) = a_0(1 - nx &+ \frac{n(n-1)}{2^2}x^2 - \frac{n(n-1)(n-2)}{2^2 \cdot 3^2}x^3 \\
&+ \frac{n(n-1)(n-2)(n-3)a_0}{2^2 \cdot 3^2 \cdot 4^2}x^4 + \cdots \\
&+ \frac{(-1)^k n!}{(k!)^2 (n-k)!}x^k + \cdots)
\end{aligned} \tag{4.7}
$$

Which is

$$
y_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k n! a_0}{(k!)^2 (n-k)!} x^k, \; k = 0, 1, 2, ... \tag{4.8}
$$

When $n = 0, 1, 2, 3, ...$ the series (4.7) ends. That is the terms after the nth term are zero. If we take a_0 to be $k!$ then we get polynomials. We call these polynomials Laguerre polyomials. The Laguerre Polynomials are

$$\begin{aligned}
L_0(x) &= 1 \\
L_1(x) &= -x + 1 \\
L_2(x) &= x^2 - 4x + 2 \\
L_3(x) &= -x^3 + 9x^2 - 18x + 6 \\
L_4(x) &= x^4 - 16x^3 + 72x^2 - 96x + 24 \\
&\;\;. \\
&\;\;. \\
&\;\;. \\
L_n(x) &= \sum_{k=0}^{n} \frac{(-1)^k (n!)^2}{(k!)^2 (n-k)!} x^k
\end{aligned} \tag{4.9}$$

The Laguerre equation is a Sturm-Liouville equation. By showing that it is one we will show that the Laguerre polynomials are orthogonal. By doing so we will be able to express more complicated functions with these polynomials. The polynomials are said to be orthogonal with respect to a weight function.

Equation (4.1) can be written in this form, but first we must multiply equation (4.1) by $r(x)$ and solve for $r(x)$.

$$r(x)xy'' + r(x)(1-x)y' + nr(x)y = 0. \tag{4.10}$$

We see by comparing (4.10) to equation (4.1) that

$$p(x) = xr(x) \tag{4.11}$$

$$p'(x) = r(x)(1-x) \tag{4.12}$$

We take the derivative of (4.11) and we obtain

$$p'(x) = xr'(x) + r(x) \tag{4.13}$$

Setting equation (4.12) equal to (4.13) and solving we get

$$\begin{aligned} xr'(x) + r(x) &= r(x) - xr(x) \\ xr'(x) + xr(x) &= 0 \\ r'(x) + r(x) &= 0 \end{aligned}$$

Therefore

$$r(x) = ce^{-x}.$$

Equation (4.10) now becomes

$$xe^{-x}y'' + e^{-x}(1-x)y' + ne^{-x}y = 0 \tag{4.14}$$

$$(e^{-x}xy')' + ne^{-x}y = 0 \tag{4.15}$$

where

$$L = (e^{-x}xy')' \tag{4.16}$$

In order to prove that the polynomial solutions to the Laguerre equation are orthogonal we must first prove that (4.16) is self adjoint.We let y_1 and y_1 be Laguerre polynomials.

$$\begin{aligned} \langle L(y_1)|y_2\rangle &= \int_0^\infty L(y_1)y_2dx \\ &= \int_0^\infty (e^{-x}xy_1')'y_2dx \\ &= e^{-x}xy_1'y_2|_0^\infty - \int_0^\infty e^{-x}xy_1'y_2'dx \\ &= 0 - 0 - \int_0^\infty e^{-x}xy_1'y_2'dx \\ &= -\int_0^\infty e^{-x}xy_1'y_2'dx \\ &= -(y_1e^{-x}xy_2'|_0^\infty - \int_0^\infty y_1(e^{-x}xy_2')'dx) \\ &= \int_0^\infty y_1(e^{-x}xy_2')'dx \\ &= \langle y_1|L(y_2)\rangle \end{aligned}$$

We know that y_1 and y_2 are continuous at 0. We also see that the weighting function $r(x) = e^{-x}$ causes

$$\lim_{x\to\infty} e^{-x} x y_1 y_2' = 0, \lim_{x\to\infty} e^{-x} x y_1' y_2 = 0$$

$$\lim_{x\to 0} e^{-x} x y_1' y_2 = 0, \lim_{x\to 0} e^{-x} x y_1 y_2' = 0$$

Here we do not need to impose boundary conditions. Also by Lemma (2.2) we can see that the polynomial solutions to the Laguerre equations are orthogonal on the interval $[0, -\infty)$ with respect to the weight function $r(x) = e^{-x}$.

5. Hermite Polynomials

The Hermite differential equation is

$$y'' - 2xy' + 2ny = 0 n \in \mathbb{R}, x \in (-\infty, \infty) \tag{5.1}$$

We know that x=0 is an ordinary point of equation (5.1). We may use the power series method to find the polynomial solutions.

$$\begin{aligned} y &= \sum_{k=0}^{\infty} a_k x^k \\ y' &= \sum_{k=0}^{\infty} a_k k x^{k-1} = \sum_{k=0}^{\infty} a_{k+1}(k+1)x^k \\ y'' &= \sum_{k=0}^{\infty} a_{k+1}(k+1)x^{k-1} = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k \end{aligned}$$

By substitution,

$$\sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k - 2x\sum_{k=0}^{\infty} a_{k+1}(k+1)x^k + 2n\sum_{k=0}^{\infty} a_k x^k = 0$$

$$\sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k - 2\sum_{k=0}^{\infty} a_{k+1}(k+1)x^{k+1} + \sum_{k=0}^{\infty} 2n a_k x^k = 0$$

$$\sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k + \sum_{k=0}^{\infty} 2n a_k x^k - 2\sum_{k=0}^{\infty} a_k k x^k = 0$$

$$\sum_{k=0}^{\infty} [a_{k+2}(k+2)(k+1)x^k + 2a_k(n-k)]x^k = 0$$

We obtain the recurrence realation by seting the coefficients equal to zero. This recurrence relation is

$$a_{k+2} = \frac{-2(n-k)}{(k+2)(k+1)} a_k$$

Remark. We are looking for polynomial solutions. If we assume our solution has degree L then

$$\sum_{k=0}^{L} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots + a_L + 0x^{L+1} + 0x^{L+2} + \cdots$$

Where all the terms following a_L will be zero, while $a_L \neq 0$. By our recursion formula, we know

$$\begin{aligned} a_{L+2} = a_L & \frac{-2(n-L)}{(L+2)(L+1)} \\ & \frac{-2(n-L)}{(L+2)(L+1)} = 0 \\ & -2(n-L) = 0 \\ & n - L = 0 \\ & n = L \end{aligned}$$

So $L = n$ is our solution because all terms after $n+1$ are zero. Therefore the degree of our polynomial solution is n where n is an integer.So we can rewrite a polynomial of degree L as

$$\sum_{k=0}^{n} a_k x^k$$

where n must be an integer. This means we can find polynomial solutions where a_k is determined by the recurrence relation above.

After evaluating the Hermite recurrence relation we find that the polynomials solutions are

$$\begin{aligned}
&P_0(x) = 1\\
&P_1(x) = 2x\\
&P_2(x) = 4x^2 - 2\\
&P_3(x) = 8x^3 - 12x\\
&P_4(x) = 16x^4 - 48x^2 + 12\\
&.\\
&.\\
&.
\end{aligned}$$

In order to show that the polynomial solutions are orthogonal we must put the Hermite equation into the Sturm-Liouville form by finding some $r(x)$ by which we will multiply the equation (5.1). As in the case of the Laguerre equation we solve a differential equation of $r(x)$ to get $r(x) = e^{-x^2}$. After multiplying equation (5.1) by $r(x)$ we get

$$\begin{aligned}
&(e^{-x^2}y')' + 2ne^{-x^2}y = 0\\
&e^{-x^2}y'' - 2xe^{-x^2}y' + 2ne^{-x^2}y = 0
\end{aligned}$$

We find the linear operator in our case to be $Ly = (e^{-x^2}y')'$ and $\lambda r(x) = 2ne^{-x^2}y$. Now, we want L to be self-adjoint for any two polynomial solutions f and g, $f \neq g$, on the interval $C(-\infty, \infty)$ so we want

$$\begin{aligned}
&\langle Lf|g\rangle = \langle f|Lg\rangle\\
&\langle Lf|g\rangle - \langle f|Lg\rangle = 0
\end{aligned}$$

So

$$\langle Lf|g\rangle - \langle f|Lg\rangle = \int_{-\infty}^{\infty} Lf(x)g(x) - f(x)Lg(x)dx = 0$$

For Hermite solutions, $Ly = [e^{-x^2}y']'$, so we want to place restrictions on f and g so that

$$\int_{-\infty}^{\infty} Lf(x)g(x) - f(x)Lg(x)dx = 0$$

From the Sturm-Liouville theory we get

$$\int_{-\infty}^{\infty} (e^{-x^2} f'(x))' g(x) - f(x)(e^{-x^2} g'(x))' dx$$
$$= \int_{-\infty}^{\infty} [(e^{-x^2})(f'(x)g(x) - g'(x)f(x))]' dx$$

After further evaluation as in the Legendre case we obtain

$$\lim_{a\to-\infty} [(e^{-x^2})(f'(x)g(x) - g'(x)f(x))]_a^0$$
$$+ \lim_{b\to\infty} [(e^{-x^2})(f'(x)g(x) - g'(x)f(x))]_0^b$$

We want

$$\lim_{x\to\pm\infty} e^{-x^2} f(x)g'(x) = 0$$

for all $f, g \in BC^2(-\infty, \infty)$. So we impose the following conditions on the space of functions we consider

$$\lim_{x\to\pm\infty} e^{-x^2/2} h(x) = 0$$

and

$$\lim_{x\to\pm\infty} e^{-x^2/2} h'(x) = 0$$

for all $h \in C^2(-\infty, \infty)$. The Hermite polynomials are orthogonal on the interval $(-\infty, \infty)$ with respect to the weight function $r(x) = e^{-x^2}$ by Lemma (2.2).

6. Acknowledgements

We would like to thank Dr. Davidson and Matthew Dawson for their patience and wisdom through this project.

References

[1] Tolstov, G.P., *Fourier Series*, Dover Publications, New York, 1976, pg. 1–65.
[2] Morris, T., Harry P., *Ordinary Differential Equations*, Dover Publications, New York,1985, pg.531-630.
[3] Goode, W.S., Annin, S.A., *Differential Equations and Linear Algebra* Pearson Education, Upper Saddle River, 2007, pg.671-722.
[4] Adkins, W.A., Davidson, M.G., *Ordinary Differential Equations*, Springer, Berlin, 2009

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