Jordan Decomposition

via the $\mathcal{Z}$-transform

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Motivation

In mathematics, one often tries to take a complicated object, and break it down into simpler pieces.
In our paper, we look at complex-valued matrices, and try to write them as a sum of matrices that are simple to work with, and have nice properties.

Example

$$A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$$

$$A = (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}$$

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In general:

$$A = \sum_{i=1}^{r} \lambda_i P_i + N_i$$

- λ_i - eigenvalues
- P_i - projections
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$\mathcal{Z}$-transform

Definition

Let $y(k)$ be a sequence of complex numbers. We define the $\mathcal{Z}$-transform of y to be the function $\mathcal{Z}\{y\}(z)$, where z is a complex variable, by the following formula:
$\mathcal{Z}\{y\}(z) = \sum_{k=0}^{\infty} \frac{y(k)}{z^k}$

Example of the $\mathcal{Z}-transform$

Suppose a is a non-zero complex number, and $y(k) = a^k$. We will calculate $\mathcal{Z}\{y\}(z)$

$$\begin{aligned} \mathcal{Z}\{y\}(z) &= \sum_{k=0}^{\infty} \frac{a^k}{z^k} \\ &= \sum_{k=0}^{\infty} \left(\frac{a}{z}\right)^k. \end{aligned}$$

This is a geometric series, so the sum is

$$\frac{1}{1-\frac{a}{z}} = \frac{z}{z-a}.$$

An Important Class of Functions

Definition

Let $k, n \in \mathbb{N}$. Then we define

$$k^{\underline{n}} = k(k-1)(k-2)\cdots(k-n+1).$$

This is called the falling factorial function.

Let us take a look at an example to illustrate how falling factorials work.

Example

We will compute $6^{\underline{4}}$. Using the definition above, we see that

$$6^{\underline{4}} = 6(6-1)(6-2)(6-3) = 6(5)(4)(3) = 360.$$

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Let $a \in \mathbb{C}$ and let $n, k \in \mathbb{N}$. Then we define

$$\varphi_{n,a}(k) = \begin{cases} \frac{a^{k-n} k^{\underline{n}}}{n!} & a \neq 0 \\ \delta_n(k) & a = 0, \end{cases}$$

where $\delta_n(k)$ *is the sequence which is* 0 *for all* $k \neq n$ *and* $\delta_n(n) = 1$*.*

With this definition, we get that $\mathcal{Z}\{\varphi_{n,a}(k)\}(z) = \frac{z}{(z-a)^{n+1}}$.

These functions are very important for the Jordan Decomposition. Note that the set $\{\varphi_{n,a}(k)\}$ is a linearly independent set of functions.

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An Important Class of Functions

Since these $\varphi_{n,a}(k)$ functions are so important to everything that follows, we will compute a few examples of them below.

Example

$$\varphi_{0,2}(k) = 2^k = (1, 2, 4, 8, 16, ...)$$

$$\varphi_{1,2}(k) = 2^{k-1}k = (0, 1, 4, 12, 32, ...)$$

$$\varphi_{2,0}(k) = \delta_2(k) = (0, 0, 1, 0, 0, 0, 0...)$$

Matrix Decomposition

Let A be an $n \times n$ matrix over the complex numbers with characteristic polynomial $c_A(z) = (z - \lambda_1)^{m_1} \cdots (z - \lambda_r)^{m_r}$. Then, there exists a decomposition

$$A^k = \sum_{i=1}^{r} \lambda_i^k P_i + \sum_{q=1}^{m_i - 1} N_i^q \varphi_{q,\lambda_i}(k).$$

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$$A^k = \sum_{i=1}^{r} \lambda_i^k P_i + \sum_{q=1}^{m_i - 1} N_i^q \varphi_{q,\lambda_i}(k)$$

has many nice properties, such as:

1. P_i is a projection
2. N_i is a nilpotent matrix
3. $P_i N_j = N_j P_i = \begin{cases} N_i & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$

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Matrix Decomposition

This decomposition is called the Jordan Decomposition of the matrix.

In this paper, our goal was to show that every complex-valued matrix can be written in this way.

Matrix Decomposition

It is important that the matrix we work with be over the complex numbers, because the complex numbers are algebraically closed.
If we chose a real-valued matrix, then our characteristic polynomial won't necessarily have a root.
We need for our matrix to have an eigenvalue to do our work, which always happens over the complex numbers.

Main Results

There is an interesting result when closely studying the equation

$$A^{k+l} = A^k A^l.$$

In Tsai's paper [T12], one result is that you can write the matrix power A^k as

$$A^k = \sum_{r=1}^{R} \sum_{m=0}^{M_r-1} B_{r,m} \varphi_{m,a_r}(k).$$

Main Results

By writing both sides as Tsai's summation decomposition, as well as identical sums we arrive at

$$A^{k+l} = \sum_{i=1}^{r}\sum_{a=0}^{\infty}\sum_{b=1}^{r}\sum_{j=0}^{\infty} M_{i,j+a}\delta_i(b)\varphi_{j,\lambda_b}(k)\varphi_{a,\lambda_i}(l)$$

$$= \sum_{i=1}^{r}\sum_{a=0}^{\infty}\sum_{b=1}^{r}\sum_{j=0}^{\infty} M_{i,a}M_{b,j}\varphi_{j,\lambda_b}(k)\varphi_{a,\lambda_i}(l) = A^k A^l.$$

Main Results

Invoking the linear independence of the φ functions, we have a collection of equations, one for each i and a. Therefore,

$$\sum_{b=1}^{r}\sum_{j=0}^{\infty} M_{i,j+a}\delta_i(b)\varphi_{j,\lambda_b}(k) = \sum_{b=1}^{r}\sum_{j=0}^{\infty} M_{i,a}M_{b,j}\varphi_{j,\lambda_b}(k),$$

and both sides still have a $\varphi_{j,\lambda_b}(k)$ in common.

Main Results

Again, using the linear independence of the φ functions, we know that the coefficients (matrices) are equal, thus

$$M_{i,j+a}\delta_i(b) = M_{i,a}M_{b,j}.$$

If we let $P_i = M_{i,0}$, we can see that $P_i^2 = P_i$, so P_i is a projection.
This is one of the properties we set out to show, and the others can be shown in a similar way.

Example

Example

Let the matrix

$$A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix},$$

where A^k can be represented as

$$A^k = \sum_{r=1}^{R} \sum_{m=0}^{M_r-1} B_{r,m} \varphi_{m,a_r}(k).$$

Example

Example

We must first find the characteristic polynomial:

$$\begin{aligned} c_A(z) &= \det(zI - A) \\ &= \begin{bmatrix} z-1 & -4 \\ 1 & z+3 \end{bmatrix} \\ &= (z-1)(z+3) - (-4) \\ &= z^2 + 2z + 1 \\ &= (z+1)^2. \end{aligned}$$

Example

Example

From the characteristic polynomial, we can find the eigenvalues a_r and the multiplicities M_r. In this example there is only one eigenvalue $a_1 = -1$ and its multiplicity is $M_1 = 2$, so the A^k equation becomes

$$A^k = \sum_{r=1}^{1} \sum_{m=0}^{1} B_{r,m} \varphi_{m,a_r}(k)$$

$$= B_{1,0}\varphi_{0,-1}(k) + B_{1,1}\varphi_{1,-1}(k).$$

Example

Example

Let us define $M = B_{1,0}$ and $N = B_{1,1}$ for simplicity, so A^k and A^ℓ become

$$A^k = M\varphi_{0,-1}(k) + N\varphi_{1,-1}(k)$$

$$A^\ell = M\varphi_{0,-1}(\ell) + N\varphi_{1,-1}(\ell).$$

Similarly,

$$A^{k+\ell} = M\varphi_{0,-1}(k+\ell) + N\varphi_{1,-1}(k+\ell).$$

Example

Example

Now we can substitute these into the equation

$$A^k A^\ell = A^{k+\ell}$$

to get:

$$[M\varphi_{0,-1}(k) + N\varphi_{1,-1}(k)][M\varphi_{0,-1}(\ell) + N\varphi_{1,-1}(\ell)]$$

$$= M\varphi_{0,-1}(k+\ell) + N\varphi_{1,-1}(k+\ell).$$

Example

Example

After expanding and simplifying, the equation can be written as

$$M(-1)^{k+\ell} + NN(k+\ell)(-1)^{k+\ell-1}$$

$$= M^2(-1)^{k+\ell} + MN\ell(-1)^{k+\ell-1} + NMk(-1)^{k+\ell-1} + N^2k\ell(-1)^{k+\ell-2}.$$

Example

Example

Divide both sides of the equation by $(-1)^{k+\ell}$ to get

$$M - N(k+\ell) = M^2 - MN\ell - NMk + N^2kl.$$

Let $k = 0, \ell = 0$. Then we get that $M = M^2$, so M is a projection.

Example

Example

We can then subtract M from both sides of the equation to get

$$N(k+\ell) = MN\ell + NMk - N^2 kl.$$

Let $k = 0, \ell = 1$. Then we get that $N = MN$. We can make a similar choice to see that $N = NM$. Therefore our equation becomes

$$N(k+\ell) = Nk + N\ell - N^2 k\ell.$$

This shows us that $N^2 = 0$.

Example

Example

From this example, we have verified the following properties:

$M^2 = M$

$MN = NM = N$

$N^2 = 0.$

Hence, M is a projection and N is nilpotent.

Summary

- Every matrix has a Jordan decomposition, made up of projections and nilpotents.
- Projections and nilpotents have many properties.
- Using the $\mathcal{Z}$ transform, we build the φ function.
- Using the lin. independance of the φ functions, and the fact that $A^{k+\ell} = A^k A^\ell$, we arrive at these properties.
- For a matrix to any power, we can easilly express it as a sum of projections and nilpotents.

Thank You

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For Further Reading I

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