Bessel's Function

A Touch of Magic

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Outline

General Form

Bessel's differential equation is

$$x^2 y'' + xy' + (x^2 - n^2)y = 0$$

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The zeros are j_n and y_n.

Bessel Functions of Order Zero

/Bessel/j0.pdf

Figure: Bessel Function of the First Kind, J_0

Bessel Functions of Order Zero

/Bessel/y0.pdf

Figure: Bessel Function of the Second Kind, Y_0

Key Terms

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- Superposition

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- $u_t(r, 0) = 0$
- $u(1, t) = 0$

Separation of Variables

We have

$$u(r, t) = R(r)T(t)$$

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- $R(r) = c_3 J_0(\alpha r) + c_4 Y_0(\alpha r)$

Evaluation

Using initial and boundary conditions, we have

$$u_n(r,t) = A_n J_0(j_n r)\cos(j_n t)$$

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General solution

$$u(r,t) = \sum_{n=1}^{\infty} A_n J_0(j_n r)\cos(j_n t)$$

Amplitude

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$$A_n = \frac{\int_0^1 rJ_0(j_n r) f(r) dr}{\int_0^1 rJ_0(j_n r) J_0(j_n r) dr}$$

Frequencies

Fundamental pitch	$\frac{j_1}{2\pi}$
First overtone	$\frac{j_2}{2\pi}$
Second overtone	$\frac{j_3}{2\pi}$

Orthogonality Property of Bessel Functions

/Bessel/jnspdf.pdf

Figure: Bessel Functions of the First Kind

Problems in Mathematical Physics

- PDE's model physical phenomena.
- Example: Steady Temperatures in Circular Cylinder (Laplacian in Cylindrical Coordinates).
- Example: The Vibrating Drumhead (Wave Equation in Polar Coordinates).

Methods of Solution

- PDE's are difficult to solve.
- Fourier's Method: Linear and homogeneous PDE's with homogeneous boundary conditions.
- Also known as Separation of Variables.

Fourier's Method: PDE $\longrightarrow$ ODE's

- PDE: Wave Equation in Polar Coordinates
- Apply Fourier's Method
- Two second order ODE's
  - Simple Harmonic Motion $T'' + \mu T = 0$
  - Bessel's Equation $R'' + \frac{1}{r}R' + R = 0$

Orthogonal Functions

- Analysis of solutions to ODE's
- Underlying Theme: Orthogonal Functions
- Examples:
  - Sine and Cosine Functions
  - Legendre Polynomials (Special Function)
  - Bessel Functions (A "Very" Special Function)

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- Define $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i$
- $\mathbf{x}$ and $\mathbf{y}$ are orthogonal when $\sum_{i=1}^{n} x_i y_i = 0$

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Example: Simple Harmonic Motion

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- Solutions are $\sin(nx)$ and $\cos(nx)$
- Easy to show that $\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,dx = 0$ for any $n, m \in \mathbb{Z}$

Example: Legendre Polynomials

- It was shown that the Legendre Polynomials satisfy $\int_{-1}^{1} P_n(x)P_m(x)\,dx = 0$ for $n, m \in \mathbb{Z}, n \neq m$

Example: Bessel Functions

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- Orthogonality property of $J_n(\lambda x)$ and $J_n(\mu x)$
- Bessel Functions of the First Kind of Order n
- λ and μ are distinct positive roots of $J_n(x) = 0$
- Will show: $\int_0^1 xJ_n(\lambda x)J_n(\mu x)\,dx = 0$

Theorem

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If λ and μ are distinct positive roots of $J_n(x) = 0$ then

$$\int_0^1 xJ_n(\lambda x)J_n(\mu x)\,dx = \begin{cases} 0, & \text{if } \lambda \neq \mu \\ \frac{1}{2}J_{n+1}^2(\lambda), & \text{if } \lambda = \mu \end{cases}$$

Proof

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Suppose $\lambda \neq \mu$, then λ and μ are distinct positive roots of $J_n(x) = 0$. Since $J_n(\lambda x)$ and $J_n(\mu x)$ are solutions of the Bessel equation in parametric form, we can write

$$x^2 J_n''(\lambda x) + x J_n'(\lambda x) + (\lambda^2 x^2 - n^2) J_n(\lambda x) = 0 \qquad (1)$$

and

$$x^2 J_n''(\mu x) + x J_n'(\mu x) + (\mu^2 x^2 - n^2) J_n(\mu x) = 0 \qquad (2)$$

Equations (1) and (2) may be written in the form

Proof

$$x\frac{d}{dx}\left[x\frac{d}{dx}J_n(\lambda x)\right]+(\lambda^2x^2-n^2)J_n(\lambda x)=0 \qquad (3)$$

and

$$x\frac{d}{dx}\left[x\frac{d}{dx}J_n(\mu x)\right]+(\mu^2x^2-n^2)J_n(\mu x)=0 \qquad (4)$$

Multiplying (3) by $\frac{J_n(\mu x)}{x}$ and (4) by $\frac{J_n(\lambda x)}{x}$ we get

Proof

$$J_n(\mu x)\frac{d}{dx}\left[x\frac{d}{dx}J_n(\lambda x)\right] + \frac{1}{x}(\lambda^2 x^2 - n^2)J_n(\lambda x)J_n(\mu x) = 0 \quad (5)$$

and

$$J_n(\lambda x)\frac{d}{dx}\left[x\frac{d}{dx}J_n(\mu x)\right] + \frac{1}{x}(\mu^2 x^2 - n^2)J_n(\mu x)J_n(\lambda x) = 0 \quad (6)$$

Then subtracting, (5) - (6) we get

Proof

$$J_n(\mu x)\frac{d}{dx}\left[x\frac{d}{dx}J_n(\lambda x)\right] -$$
$$J_n(\lambda x)\frac{d}{dx}\left[x\frac{d}{dx}J_n(\mu x)\right] +$$
$$(\lambda^2 - \mu^2)xJ_n(\lambda x)J_n(\mu x) = 0 \tag{7}$$

With some more manipulation, equation (7) may be written as

Proof

$$\frac{d}{dx}\left[J_n(\mu x)x\frac{d}{dx}J_n(\lambda x)\right]-$$
$$\frac{d}{dx}\left[J_n(\lambda x)x\frac{d}{dx}J_n(\mu x)\right]+$$
$$(\lambda^2-\mu^2)xJ_n(\lambda x)J_n(\mu x)=0 \tag{8}$$

Finally integrating (8) from 0 to 1 noting that $J_n(\lambda) = J_n(\mu) = 0$, we get

Proof

$$(\lambda^2 - \mu^2) \int_0^1 x J_n(\lambda x) J_n(\mu x)\, dx = 0$$

And since $\lambda \neq \mu$, then we may divide to get the desired result

$$\int_0^1 x J_n(\lambda x) J_n(\mu x)\, dx = 0 \tag{9}$$

□

Coefficients

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If λ and μ are distinct positive roots of $J_n(x) = 0$ then

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- $\int_0^1 xJ_n(\lambda x)J_n(\lambda x)\,dx = \frac{1}{2}J_{n+1}^2(\lambda) \neq 0$

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- $\int_0^1 x J_n(\lambda x) J_n(\lambda x)\, dx = \frac{1}{2} J_{n+1}^2(\lambda) \neq 0$
- $\int_0^1 r J_0(j_n r) J_0(j_n r)\, dr = \frac{1}{2} J_1^2(j_n) \neq 0$

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- $\int_0^1 xJ_n(\lambda x)J_n(\lambda x)\,dx = \frac{1}{2}J_{n+1}^2(\lambda) \neq 0$
- $\int_0^1 rJ_0(j_n r)J_0(j_n r)\,dr = \frac{1}{2}J_1^2(j_n) \neq 0$
- $A_n = \frac{\int_0^1 rJ_0(j_n r)f(r)\,dr}{\int_0^1 rJ_0(j_n r)J_0(j_n r)\,dr}$

Thank You!