

# Bessel's Function

## A Touch of Magic

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# Outline

## Introduction

Bessel Functions

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Drum Example

## Properties

Orthogonality

# General Form

Bessel's differential equation is

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The zeros are  $j_n$  and  $y_n$ .

# Bessel Functions of Order Zero

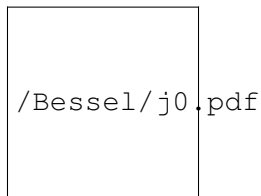


Figure: Bessel Function of the First Kind,  $J_0$

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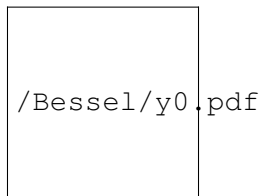


Figure: Bessel Function of the Second Kind,  $Y_0$

# Key Terms

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- ▶  $u(1, t) = 0$

# Separation of Variables

We have

$$u(r, t) = R(r)T(t)$$

- ▶  $T'' + \mu T = 0$
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- ▶  $\mu = \alpha^2 > 0$

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- ▶  $T(t) = c_1 \cos(\alpha t) + c_2 \sin(\alpha t)$
- ▶  $R(r) = c_3 J_0(\alpha r) + c_4 Y_0(\alpha r)$



# Evaluation

Using initial and boundary conditions, we have

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$$u(r, t) = \sum_{n=1}^{\infty} A_n J_0(j_n r) \cos(j_n t)$$

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$$A_n = \frac{\int_0^1 r J_0(j_n r) f(r) dr}{\int_0^1 r J_0(j_n r) J_0(j_n r) dr}$$

# Frequencies

Fundamental pitch	$\frac{j_1}{2\pi}$
First overtone	$\frac{j_2}{2\pi}$
Second overtone	$\frac{j_3}{2\pi}$

# Orthogonality Property of Bessel Functions

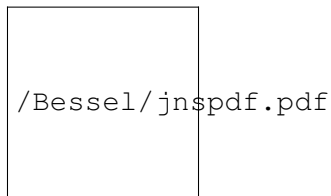


Figure: Bessel Functions of the First Kind

# Problems in Mathematical Physics

- ▶ PDE's model physical phenomena.
- ▶ Example: Steady Temperatures in Circular Cylinder (Laplacian in Cylindrical Coordinates).
- ▶ Example: The Vibrating Drumhead (Wave Equation in Polar Coordinates).



# Methods of Solution

- ▶ PDE's are difficult to solve.
- ▶ Fourier's Method: Linear and homogeneous PDE's with homogeneous boundary conditions.
- ▶ Also known as Separation of Variables.

# Fourier's Method: PDE $\longrightarrow$ ODE's

- ▶ PDE: Wave Equation in Polar Coordinates
- ▶ Apply Fourier's Method
- ▶ Two second order ODE's
  - ▶ Simple Harmonic Motion  $T'' + \mu T = 0$
  - ▶ Bessel's Equation  $R'' + \frac{1}{r}R' + R = 0$

# Orthogonal Functions

- ▶ Analysis of solutions to ODE's
- ▶ Underlying Theme: Orthogonal Functions
- ▶ Examples:
  - ▶ Sine and Cosine Functions
  - ▶ Legendre Polynomials (Special Function)
  - ▶ Bessel Functions (A "Very" Special Function)

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- ▶  $\mathbf{x}$  and  $\mathbf{y}$  are orthogonal when  $\sum_{i=1}^n x_i y_i = 0$

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- ▶ Solutions are  $\sin(nx)$  and  $\cos(nx)$
- ▶ Easy to show that  $\int_{-\pi}^{\pi} \sin(mx) \cos(nx) dx = 0$   
for any  $n, m \in \mathbb{Z}$

# Example: Legendre Polynomials

- ▶ It was shown that the Legendre Polynomials satisfy  $\int_{-1}^1 P_n(x)P_m(x) dx = 0$  for  $n, m \in \mathbb{Z}, n \neq m$

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- ▶ Bessel Functions of the First Kind of Order  $n$
- ▶  $\lambda$  and  $\mu$  are distinct positive roots of  $J_n(x) = 0$
- ▶ Will show:  $\int_0^1 x J_n(\lambda x) J_n(\mu x) dx = 0$

# Theorem

## Theorem

If  $\lambda$  and  $\mu$  are distinct positive roots of  $J_n(x) = 0$  then

$$\int_0^1 x J_n(\lambda x) J_n(\mu x) dx = \begin{cases} 0, & \text{if } \lambda \neq \mu \\ \frac{1}{2} J_{n+1}^2(\lambda), & \text{if } \lambda = \mu \end{cases}$$

# Proof

## Proof.

Suppose  $\lambda \neq \mu$ , then  $\lambda$  and  $\mu$  are distinct positive roots of  $J_n(x) = 0$ . Since  $J_n(\lambda x)$  and  $J_n(\mu x)$  are solutions of the Bessel equation in parametric form, we can write

$$x^2 J_n''(\lambda x) + x J_n'(\lambda x) + (\lambda^2 x^2 - n^2) J_n(\lambda x) = 0 \quad (1)$$

and

$$x^2 J_n''(\mu x) + x J_n'(\mu x) + (\mu^2 x^2 - n^2) J_n(\mu x) = 0 \quad (2)$$

Equations (1) and (2) may be written in the form

## Proof

$$x \frac{d}{dx} \left[ x \frac{d}{dx} J_n(\lambda x) \right] + (\lambda^2 x^2 - n^2) J_n(\lambda x) = 0 \quad (3)$$

and

$$x \frac{d}{dx} \left[ x \frac{d}{dx} J_n(\mu x) \right] + (\mu^2 x^2 - n^2) J_n(\mu x) = 0 \quad (4)$$

Multiplying (3) by  $\frac{J_n(\mu x)}{x}$  and (4) by  $\frac{J_n(\lambda x)}{x}$  we get

## Proof

$$J_n(\mu x) \frac{d}{dx} \left[ x \frac{d}{dx} J_n(\lambda x) \right] + \frac{1}{x} (\lambda^2 x^2 - n^2) J_n(\lambda x) J_n(\mu x) = 0 \quad (5)$$

and

$$J_n(\lambda x) \frac{d}{dx} \left[ x \frac{d}{dx} J_n(\mu x) \right] + \frac{1}{x} (\mu^2 x^2 - n^2) J_n(\mu x) J_n(\lambda x) = 0 \quad (6)$$

Then subtracting, (5) - (6) we get

## Proof

$$\begin{aligned} & J_n(\mu x) \frac{d}{dx} \left[ x \frac{d}{dx} J_n(\lambda x) \right] - \\ & J_n(\lambda x) \frac{d}{dx} \left[ x \frac{d}{dx} J_n(\mu x) \right] + \\ & (\lambda^2 - \mu^2) x J_n(\lambda x) J_n(\mu x) = 0 \end{aligned} \quad (7)$$

With some more manipulation, equation (7) may be written as



## Proof

$$\begin{aligned} & \frac{d}{dx} \left[ J_n(\mu x) x \frac{d}{dx} J_n(\lambda x) \right] - \\ & \frac{d}{dx} \left[ J_n(\lambda x) x \frac{d}{dx} J_n(\mu x) \right] + \\ & (\lambda^2 - \mu^2) x J_n(\lambda x) J_n(\mu x) = 0 \end{aligned} \quad (8)$$

Finally integrating (8) from 0 to 1 noting that  $J_n(\lambda) = J_n(\mu) = 0$ , we get

## Proof

$$(\lambda^2 - \mu^2) \int_0^1 x J_n(\lambda x) J_n(\mu x) dx = 0$$

And since  $\lambda \neq \mu$ , then we may divide to get the desired result

$$\int_0^1 x J_n(\lambda x) J_n(\mu x) dx = 0 \quad (9)$$



# Coefficients

## Theorem

If  $\lambda$  and  $\mu$  are distinct positive roots of  $J_n(x) = 0$  then

$$\int_0^1 x J_n(\lambda x) J_n(\mu x) dx = \begin{cases} 0, & \text{if } \lambda \neq \mu \\ \frac{1}{2} J_{n+1}^2(\lambda), & \text{if } \lambda = \mu \end{cases}$$

►  $\int_0^1 x J_n(\lambda x) J_n(\lambda x) dx = \frac{1}{2} J_{n+1}^2(\lambda) \neq 0$

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- ▶  $\int_0^1 x J_n(\lambda x) J_n(\lambda x) dx = \frac{1}{2} J_{n+1}^2(\lambda) \neq 0$
- ▶  $\int_0^1 r J_0(j_n r) J_0(j_n r) dr = \frac{1}{2} J_1^2(j_n) \neq 0$

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- ▶  $\int_0^1 r J_0(j_n r) J_0(j_n r) dr = \frac{1}{2} J_1^2(j_n) \neq 0$
- ▶  $A_n = \frac{\int_0^1 r J_0(j_n r) f(r) dr}{\int_0^1 r J_0(j_n r) J_0(j_n r) dr}$

# Thank You!