BESSEL FUNCTIONS

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ABSTRACT. We briefly address how to solve Bessel's differential equation and describe its solutions, Bessel functions. Additionally, we discuss two real-life scenarios to motivate and demonstrate the importance of Bessel functions. Finally, we discuss and prove orthogonality for Bessel functions of the first kind.

1. INTRODUCTION

The standard form for any second order homogeneous differential equation is

$$y'' + P(x)y' + Q(x)y = 0. \tag{1.1}$$

If both $P(x)$ and $Q(x)$ are analytic at x_0, meaning there is a power series, $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converging to the required function on some neighborhood of x_0, then the point x_0 is said to be ordinary, otherwise it is singular. To solve equation (1.1), we first determine whether x_0 is an ordinary or singular point. When a singular point is regular, that is, both $(x-x_0)P(x)$ and $(x-x_0)^2Q(x)$ are analytic at x_0, we can use the the method of Frobenius to solve the differential equation.

Bessel functions of order n are solutions to the second order differential equation

$$x^2y'' + xy' + (x^2 - n^2)y = 0, \tag{1.2}$$

where n is an arbitrary, constant value. We will limit our focus to values where $n \geq 0$. Equation 1.2 has two linearly independent solutions $J_n(x)$ and $Y_n(x)$ and thus a general solution

$$y = c_1J_n(x) + c_2Y_n(x).$$

We denote the Bessel function of the first kind with order n by $J_n(x)$ and the second kind by $Y_n(x)$. Similarly, we notate the zeros of these functions as j_n and y_n, respectively. Zeroth order Bessel functions of the first and second kind can be graphically represented by Figures 1 and 2, respectively.

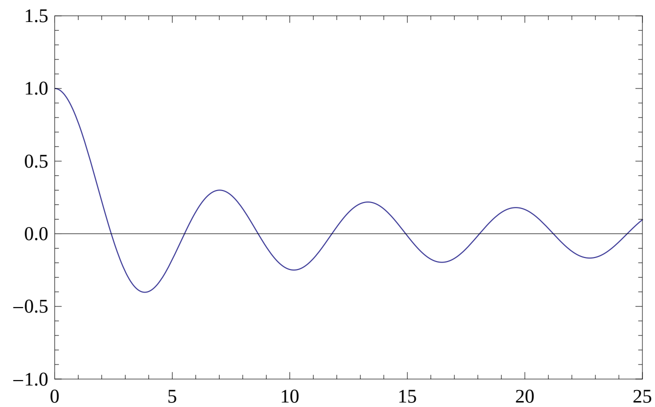

FIGURE 1. Bessel Function of the first kind, J_0

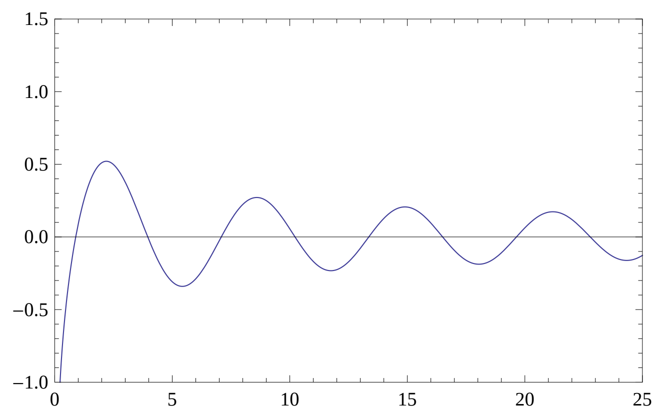

FIGURE 2. Bessel Function of the second kind, Y_0

For differential equations of the form 1.2, x_0 is regular singular. Solving the indicial equation

$$r(r-1)+a_0r+b_0r=0$$

for r gives the values we use in the Frobenius method. In this equation, a_0 and b_0 are the analytic values of $P(x)$ and $Q(x)$, in equation 1.1, respectively. The solutions to the indicial equation of 1.2 are n and $-n$ for all $n \geq 0$. The general form for $J_n(x)$ is given by

$$J_n(x)=\sum_{m=0}^{\infty}\frac{(-1)^m(\frac{x}{2})^{n+2m}}{\Gamma(m+1)\Gamma(n+m+1)},$$

where the Gamma function is defined by

$$\Gamma(p)=\int_0^{\infty}e^{-x}x^{p-1}dx$$

for $p \neq 0, -1, -2, \ldots$ and has the following properties:

(1) $\Gamma(1) = 1$
(2) $\Gamma(p+1) = p\Gamma(p)$
(3) $\Gamma(p+1) = p!$ for positive integral p.

As a result of the properties of the gamma function, $J_n(x)$ has certain properties under specific conditions as seen in the following chart.

$J_n(x)$	Condition
Even function	p even
Odd function	p odd
Complex values	$x < 0$ or $x \notin \mathbb{Z}$

Since $J_n(x)$ has complex values for $x < 0$ and non-integer values of p, we consider only values of $x > 0$ when p is not an integer.

2. Application

2.1. **Heat Diffusion.** One common application that results in a Bessel function solution is steady-state temperature in a cylinder. This particular example is a cylinder with a radius of $r = 3$ and a height of $z = 5$. The temperature is then expressed as an equation of three variables: r, θ, and z. The lateral surface and bottom face of this radially symmetric cylinder are held at temperature zero. The top face is held at an arbitrary temperature $g(r)$. Since the cylinder is radially symmetric, the temperature will never depend on θ, and all functions of θ can thus be omitted from the equation. The conditions of the aforementioned scenario can be mathematically modeled by the following boundary valued problem where $0 \leq r < 3$ and $0 < z < 5$:

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + u_{zz} = 0 \\ u(3,z) = 0 \\ u(r,0) = 0 \\ u(r,5) = g(r) \end{cases}$$

Lemma 2.1 (Separation of Variables)**.** *Suppose*

$$u(r,z) = R(r)Z(z) \neq 0; \tag{2.1}$$

then the preceeding boundary valued problem results in the following two ordinary differential equations

$$\begin{cases} Z'' - \lambda Z = 0 \\ rR'' + R' + \lambda rR = 0. \end{cases}$$

Proof. Let $u(r,z) = R(r)Z(z) \neq 0$, and observe

$$u_{rr} + \frac{1}{r}u_r + u_{zz} = R''Z + \frac{1}{r}R'Z + RZ'' = 0.$$

Dividing through by $u(r,z)$ we obtain

$$\frac{R''}{R} + \frac{\frac{1}{r}R'}{R} + \frac{Z''}{Z} = 0.$$

Since $R(r)$ and $Z(z)$ are independent from each other, they must both be some arbitrary constant, so we can equate them by

$$\frac{R''}{R} + \frac{\frac{1}{r}R'}{R} = -\frac{Z''}{Z} = -\lambda.$$

This is then decomposed into two separate second order ordinary differential equations

$$R'' + \frac{1}{r}R' + \lambda R = 0$$

and

$$Z'' - \lambda Z = 0.$$

□

The general solution for Z is

$$Z = c_1 \cosh(\alpha z) + c_2 \sinh(\alpha z) \tag{2.2}$$

when we set $\lambda = \alpha^2 > 0$. To find $c_1 = 0$, we substitute the boundary value $u(r,0) = 0$ into equation 2.2. Likewise, the general solution for R is

$$R = c_3 J_0(\alpha r) + c_4 Y_0(\alpha r).$$

In order to keep R bounded, we require $c_4 = 0$ because Y_0 approaches negative infinity as x approaches zero. The solution $\alpha_n = \frac{j_n}{3}$ is found using the boundary condition $u(3,z) = 0$. Plugging the solution back into equation 2.1 and applying the method of linearity, we have the general solution

$$u(r,z) = \sum_{n=1}^{\infty} A_n J_0\left(\frac{j_n r}{3}\right) \sinh\left(\frac{j_n z}{3}\right). \tag{2.3}$$

We find A_n to be

$$A_n = \frac{\int_0^3 r g(r) J_0(\frac{j_n r}{3}) dr}{\sinh(\frac{5 j_n}{3}) \int_0^3 r J_0(\frac{j_n r}{3}) J_0(\frac{j_n r}{3}) dr},$$

for $n = 1, 2, 3...$, using knowledge of orthogonality and substituting the boundary condition $u(r, 5) = g(r)$ into equation 2.3. This is further simplified to

$$A_n = \frac{\int_0^3 rg(r)J_0(\frac{j_n r}{3})dr}{\sinh(\frac{5j_n}{3})(\frac{9}{2})(J_1(j_n))^2}$$

for $n = 1, 2, 3...$

2.2. **Wave Propagation.** The next physical application we discuss is a radially symmetrical circular drumhead of radius $r = 1$ with fixed edges that is displaced directly in the center with an arbitrary force. The velocity of the drumhead at time $t = 0$ is zero. The initial position of the center's displacement is represented by $f(r)$. We have the boundary valued problem where $0 \leq r < 1$, $t > 0$:

$$\begin{cases} u_{tt} = u_{rr} + \frac{1}{r}u_r \\ u(r, 0) = f(r) \\ u_t(r, 0) = 0 \\ u(1, t) = 0 \end{cases}$$

Using separation of variables where

$$u(r, t) = R(r)T(t) \tag{2.4}$$

the following results:

$$\begin{cases} T'' + \mu T = 0 \\ R'' + \frac{1}{r}R' + \mu R = 0. \end{cases}$$

We disregard dampening forces and assume the drumhead will vibrate forever. The general solution

$$T(t) = c_1 \cos(\alpha t) + c_2 \sin(\alpha t) \tag{2.5}$$

results from the eigenvalue $\mu = \alpha^2 > 0$.

We substitute the initial condition $u_t(r, 0) = 0$ into this differential equation as $T'(0) = 0$. Since $\alpha^2 > 0$, we know $c_2 = 0$. The general solution for equation 2.5 becomes

$$T(t) = c_1 \cos(\alpha t).$$

We note that $r = 0$ is a regular singular point of the ordinary differential equation of $R(r)$. A slight change in variables leads to the following Bessel function

$$R(r) = c_3 J_0(\alpha r) + c_4 Y_0(\alpha r). \tag{2.6}$$

We know $R(r)$ must be bounded at 0. Observing the graphs of J_0 and Y_0 in Figure 1 and Figure 2, we have c_4, in equation 2.6, must be equal to zero. The boundary condition $R(1) = 0$ yields $\alpha_n = j_n$. Using this information we can substitute back into equation 2.4 to obtain

$$u_n(r,t) = A_n J_0(j_n r)\cos(j_n t).$$

The general solution $u(r,t)$ is the Fourier-Bessel series

$$u(r,t) = \sum_{n=1}^{\infty} A_n J_0(j_n r)\cos(j_n t). \tag{2.7}$$

The initial condition $u(r,0) = f(r)$ and knowledge of orthogonality can be used to find the amplitude of the initial displacement of the drumhead. The amplitude of the displacement, or Fourier coefficient A_n, is found by evaluating the orthogonal relationship

$$A_n = \frac{\int_0^1 rJ_0(j_n r)f(r)dr}{\int_0^1 rJ_0(j_n r)J_0(j_n r)dr}$$

which depends on the force.

From equation 2.7, the frequency, fundamental pitch of the drum, and all overtones can be found. The period is found using the parameter of $\cos(j_n t)$, and is $\frac{2\pi}{j_n}$, so the frequency is $\frac{j_n}{2\pi}$. The fundamental pitch is the frequency $\frac{j_1}{2\pi}$. All overtones are found as the frequency evaluated at each zero j_n with $n = 2, 3, 4...$ where j_2 defines the first overtone, j_3 the second, and so forth. When the ratios between overtones and the fundamental pitch are integer values, the sound is harmonious. When the relation is not an integer value, the sound is cacophonous.

3. Orthogonality

We now explore the orthogonality property of $J_n(\lambda x)$ and $J_n(\rho x)$, the Bessel functions of the first kind of order n where λ and ρ are distinct positive roots of $J_n(x) = 0$. We do this by proving

$$\int_0^1 xJ_n(\lambda x)J_n(\rho x)\,dx = 0.$$

A few remarks are in order. Recall the dot product or inner product defined for $\mathbb{R}^n$. Two vectors $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$ are said to be orthogonal if

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i y_i = 0.$$

As a generalization we define the inner product for the space of Riemann integrable functions $\mathcal{R}[a,b]$ on a closed and bounded interval by

$$\langle f, g \rangle = \int_a^b f(x)g(x)\,dx.$$

We say f, g are orthogonal if $\langle f, g\rangle = 0$. Sometimes, as in the case of Bessel functions, inner products are defined with a *weight function* $w(x)$. In this case, we say $f, g \in \mathcal{R}[a, b]$ are orthogonal if

$$\langle f, g\rangle = \int_a^b w(x) f(x) g(x)\, dx = 0.$$

In the case of Bessel functions, we have $w(x) = x$. We are now ready to prove the following theorem.

Theorem 3.1. *If λ and ρ are positive roots of $J_n(x)$, then*

$$\int_0^1 x J_n(\lambda x) J_n(\rho x)\, dx = \begin{cases} 0, & \text{if } \lambda \neq \rho \\ \frac{1}{2} J_{n+1}^2(\lambda), & \text{if } \lambda = \rho \end{cases}$$

Proof. (i) Suppose $\lambda \neq \rho$. Then λ and ρ are distinct positive roots of $J_n(x)$. Since $J_n(\lambda x)$ and $J_n(\rho x)$ are solutions of the Bessel equation in parametric form, we can write

$$x^2 J_n''(\lambda x) + x J_n'(\lambda x) + (\lambda^2 x^2 - n^2) J_n(\lambda x) = 0 \tag{3.1}$$

and

$$x^2 J_n''(\rho x) + x J_n'(\rho x) + (\rho^2 x^2 - n^2) J_n(\rho x) = 0. \tag{3.2}$$

Equations (3.1) and (3.2) may be written in the form

$$x \frac{d}{dx}\left[x \frac{d}{dx} J_n(\lambda x)\right] + (\lambda^2 x^2 - n^2) J_n(\lambda x) = 0 \tag{3.3}$$

and

$$x \frac{d}{dx}\left[x \frac{d}{dx} J_n(\rho x)\right] + (\rho^2 x^2 - n^2) J_n(\rho x) = 0. \tag{3.4}$$

Multiplying equation (3.3) by $\frac{J_n(\rho x)}{x}$ and equation (3.4) by $\frac{J_n(\lambda x)}{x}$, we obtain

$$J_n(\rho x) \frac{d}{dx}\left[x \frac{d}{dx} J_n(\lambda x)\right] + \frac{1}{x}(\lambda^2 x^2 - n^2) J_n(\lambda x) J_n(\rho x) = 0 \tag{3.5}$$

and

$$J_n(\lambda x) \frac{d}{dx}\left[x \frac{d}{dx} J_n(\rho x)\right] + \frac{1}{x}(\rho^2 x^2 - n^2) J_n(\rho x) J_n(\lambda x) = 0. \tag{3.6}$$

Then subtracting (3.6) from (3.5), we obtain

$$\begin{aligned} J_n(\rho x)\frac{d}{dx}\left[x\frac{d}{dx}J_n(\lambda x)\right] - J_n(\lambda x)\frac{d}{dx}\left[x\frac{d}{dx}J_n(\rho x)\right] + \\ (\lambda^2-\rho^2)xJ_n(\lambda x)J_n(\rho x) = 0. \end{aligned} \tag{3.7}$$

We rewrite (3.7) as

$$\begin{aligned} \frac{d}{dx}\left[J_n(\rho x)x\frac{d}{dx}J_n(\lambda x)\right] - \left[\frac{d}{dx}J_n(\rho x)\right]x\frac{d}{dx}J_n(\lambda x) \\ -\frac{d}{dx}\left[J_n(\lambda x)x\frac{d}{dx}J_n(\rho x)\right] + \left[\frac{d}{dx}J_n(\lambda x)\right]x\frac{d}{dx}J_n(\rho x) \\ +(\lambda^2-\rho^2)xJ_n(\lambda x)J_n(\rho x) = 0 \end{aligned} \tag{3.8}$$

and simplify to obtain

$$\begin{aligned} \frac{d}{dx}\left[J_n(\rho x)x\frac{d}{dx}J_n(\lambda x)\right] - \frac{d}{dx}\left[J_n(\lambda x)x\frac{d}{dx}J_n(\rho x)\right] + \\ (\lambda^2-\rho^2)xJ_n(\lambda x)J_n(\rho x) = 0. \end{aligned} \tag{3.9}$$

Now integrating equation (3.9) with respect to x from 0 to 1, noting that $J_n(\lambda) = J_n(\rho) = 0$, we get

$$(\lambda^2-\rho^2)\int_0^1 xJ_n(\lambda x)J_n(\rho x)\,dx = 0. \tag{3.10}$$

Since $\lambda \neq \rho$, we have

$$\int_0^1 xJ_n(\lambda x)J_n(\rho x)\,dx = 0 \tag{3.11}$$

as desired completing the first part of the proof.

(ii) To prove the second part, let $u = J_n(\lambda x)$, then

$$x^2u'' + xu' + (\lambda^2x^2 - n^2)u = 0.$$

Multiplying by $2u'$, we get

$$2x^2u''u' + xu'^2 + (\lambda^2x^2 - n^2)uu' = 0.$$

We can rewrite this equation as

$$\frac{d}{dx}\left[x^2u'^2 - n^2u^2 + \lambda^2x^2u^2\right] - 2\lambda^2xu^2 = 0.$$

Integrating with respect to x from 0 to 1, we get

$$\left[x^2u'^2 - n^2u^2 + \lambda^2x^2u^2\right]_0^1 - 2\lambda^2\int_0^1 xu^2\,dx = 0$$

or

$$\left[x^2\left(\frac{d}{dx}J_n(\lambda x)\right)^2 - n^2J_n^2(\lambda x) + \lambda^2x^2J_n^2(\lambda x)\right]_0^1 - 2\lambda^2\int_0^1 xJ_n^2(\lambda x)\,dx = 0.$$

But $J_n(\lambda) = 0$ and $nJ_n(0) = 0$, so

$$\left[\left(\frac{d}{dx}J_n(\lambda x)\right)^2\right]\Bigg|_{x=1} - 2\lambda^2\int_0^1 xJ_n^2(\lambda x)\,dx = 0.$$

Hence

$$\int_0^1 xJ_n^2(\lambda x)\,dx = \frac{1}{2\lambda^2}\left[\left(\frac{d}{dx}J_n(\lambda x)\right)^2\right]\Bigg|_{x=1}, \tag{3.12}$$

and from the recurrence relations of Bessel Functions, we have

$$\frac{d}{d(\lambda x)}J_n(\lambda x) = \frac{n}{\lambda x}J_n(\lambda x) - J_{n+1}(\lambda x)$$

or

$$\frac{d}{dx}J_n(\lambda x) = \frac{n}{x}J_n(\lambda x) - \lambda J_{n+1}(\lambda x).$$

Then equation (3.12) becomes

$$\begin{aligned}\int_0^1 xJ_n^2(\lambda x)\,dx &= \frac{1}{2\lambda^2}\left[\left(\frac{n}{x}J_n(\lambda x) - \lambda J_{n+1}(\lambda x)\right)^2\right]\Bigg|_{x=1}\\ &= \frac{1}{2}J_{n+1}^2(\lambda)\end{aligned}$$

as desired, completing the second part of the proof. □

Remark 3.2. The proof of the second part of Theorem 3.1 justifies division by $\int_0^1 rJ_0(j_nr)J_0(j_nr)\,dr$ in the computation of the Fourier Coefficient

$$A_n = \frac{\int_0^1 rJ_0(j_nr)f(r)\,dr}{\int_0^1 rJ_0(j_nr)J_0(j_nr)\,dr}$$

for the wave equation in polar coordinates.

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References

[1] Benson, Dave, "The Drum." Music: A Mathematical Offering. 103-108. N.p., n.d. Web. 9 July 2010. http://www.maths.abdn.ac.uk/ bensondj/html/music.pdf.
[2] Boyce, W.E. and DiPrima, R.C., *Elementary Differential Equations and Boundary Value Problems*, John Wiley & Sons, New York, 1997.
[3] Coddington, E.A., *An Introduction to Ordinary Differential Equations*, Dover Publications, New York, 1989.
[4] El Attar, R., *Bessel and Related Functions*, Lulu Press, North Carolina, 2007.
[5] Jackson, D., *Fourier Series and Orthogonal Polynomials*, Dover Publications, New York, 2004.
[6] Tolstov, G. P., *Fourier Series*, Dover Publications, New York, 1976.

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