Abstract

The goal of this project is to come up with a dynimatical system that represents an eco-system of three populations. Then apply theorems and methods learned in class to detemine under which contiditons this population can co-exsist.

A Predator-Prey Math Biology Model

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After much trial and error, We came up with this model. Which is a two predator and one prey model.

$$dx/dt = x(\beta z - x)$$

$$dy/dt = y(\gamma z - \alpha)$$

$$dz/dt = z(\theta - \phi x - \eta y)$$

$$(\alpha, \beta, \gamma, \theta, \phi, \eta > 0)$$

- x represents the population of the first predator.
- y represents the population of the second predator.
- z represents the population of the prey.
- βz represents the per captia gain to the first predator.
- γz represents the per captia gain to the second predator.
- α represents the rate for which the second predator is harvested.
- θ represents the birth rate constant of the prey z.
- ϕx represents the per captia loss of the prey due to the first predator.
- ηy represents the per captia loss of the prey due to the second predator.

In dx/dt we can see that as the population of x increases its death rate also increases. Thus if $betaz < x$, dx/dt will have a carrying capcity.
If In dy/dt, $gammaz < alpha$, then the second preadator will die off. And in the absence of predators dz/dt will grow with out bound. Which brings us to the main focus of this project. Under what conditions can this model be a viable stable eco-system.

First we find the critical points for our system.

1. (0,0,0)
2. $(0, \frac{\alpha}{\gamma}, \frac{\theta}{\eta})$
3. $(\frac{\theta}{\phi}, 0, \frac{\theta}{\beta\phi})$
4. $(\frac{\beta\alpha}{\gamma}, \frac{\theta - \phi\beta\frac{\alpha}{\gamma}}{\eta}, \frac{\alpha}{\gamma})$

Then We can use a Jacobian Matrix to linearize our system.

$$J = \begin{pmatrix} f_x(x,y,z) & f_y(x,y,z) & f_z(x,y,z) \\ g_x(x,y,z) & g_y(x,y,z) & g_z(x,y,z) \\ h_x(x,y,z) & h_y(x,y,z) & h_z(x,y,z) \end{pmatrix}$$

Now by plugging in our Critical points into the Jacobian matrix, When can take its eignvalues to prove stabily.

Lemma

The critical point is stable if the eigenvuales of its matrix have all negative parts or negative real parts. If one of its eigenvalues is positive the point is unstable.

First we will look at the origin, (0,0,0). Plugging in our values for x, y, and z, we can attain our H_2 matrix.

$$J(0,0,0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

To find the roots of the system, it is easiest to take the determinant of a $\lambda * I - H_2$, which yields:

$$\lambda * I - J(0,0,0) = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda + \alpha & 0 \\ 0 & 0 & \lambda - \theta \end{pmatrix}$$

This gives us our roots to this H_2 matrix fairly readily as the determinant yeilds the equation

$$\lambda(\lambda + \alpha)(\lambda - \theta) = 0$$

Thus $\lambda_1 = 0$, $\lambda_2 = -\alpha$, and $\lambda_3 = \theta$. Since $\lambda_1 \not< 0$ and $\lambda_3 \not< 0$

Next we will analyze the critical point $(0, \frac{\alpha}{\gamma}, \frac{\theta}{\eta})$.

$$J(0, \frac{\alpha}{\gamma}, \frac{\theta}{\eta}) = \begin{pmatrix} \frac{\beta\theta}{\eta} & 0 & 0 \\ 0 & \frac{\gamma\theta}{\eta} & \alpha \\ \frac{-\phi\theta}{\eta} & -\theta & \theta - \frac{\eta\alpha}{\gamma} \end{pmatrix}$$

To find the roots of the system, it is easiest to take the determinant of a $\lambda * I - H_2$, which yields:

$$\lambda * I - J(0, \frac{\alpha}{\gamma}, \frac{\theta}{\eta}) = \begin{pmatrix} \lambda - \frac{\beta\theta}{\eta} & 0 & 0 \\ 0 & \lambda - \frac{\gamma\theta}{\eta} & -\alpha \\ \frac{\phi\theta}{\eta} & \theta & \lambda - \theta + \frac{\eta\alpha}{\gamma} \end{pmatrix}$$

Which yeilds a $P(\lambda)$ that looks like this:

$$(\lambda - \frac{\beta\theta}{\eta})((\lambda - \frac{\gamma\theta}{\eta})(\lambda - \theta + \frac{\eta\alpha}{\gamma}) + \alpha\theta) = 0$$

and reduces farther to:

$$(\lambda - \frac{\beta\theta}{\eta})(\lambda^2 + \lambda(\frac{\eta}{\gamma} - \theta - \frac{\gamma\theta}{\eta}) + (\frac{\gamma\theta^2}{\eta}) = 0$$

Thus $\lambda_1 = \frac{\beta\theta}{\eta}$, $\lambda_{2,3} = \frac{\theta + \frac{\gamma\theta}{\eta} - \frac{\eta\alpha}{\gamma} \pm \sqrt{(\frac{\eta\alpha}{\gamma} - \theta - \frac{\gamma\theta}{\eta})^2 - 4\frac{\gamma\theta^2}{\eta}}}{2}$. Since $\lambda_1 \nless 0$, Next we will analyze the critical point $(\frac{\theta}{\phi}, 0, \frac{\theta}{\beta\phi})$. Plugging in our values for x, y, and z, we can attain our H_2 matrix.

$$J(\frac{\theta}{\phi}, 0, \frac{\theta}{\beta\phi}) = \begin{pmatrix} -\frac{\theta}{\phi} & 0 & \frac{\theta\beta}{\phi} \\ 0 & \frac{\gamma\theta}{\beta\phi} - \alpha & 0 \\ -\frac{\phi}{\beta} & -\frac{\eta\theta}{\beta\phi} & 0 \end{pmatrix}$$

To find the roots of the system, it is easiest to take the determinant of a $\lambda * I - H_2$, which yields:

$$\lambda * I - J(\frac{\theta}{\phi}, 0, \frac{\theta}{\beta\phi}) = \begin{pmatrix} \lambda + \frac{\theta}{\phi} & 0 & -\frac{\theta\beta}{\phi} \\ 0 & \lambda - \frac{\gamma\theta}{\beta\phi} + \alpha & 0 \\ \frac{\phi}{\beta} & \frac{\eta\theta}{\beta\phi} & \lambda \end{pmatrix}$$

Which yeilds a $P(\lambda)$ that looks like this:

$$\lambda(\lambda + \frac{\theta}{\phi})(\lambda + \alpha - \frac{\gamma\theta}{\beta\phi}) - \frac{\theta\beta}{\phi} * \frac{\theta}{\beta}(\lambda + \alpha - \frac{\gamma\theta}{\beta\gamma}) = 0$$

Thus $\lambda_1 = -\frac{-\phi\beta - \phi\beta\alpha + \gamma\theta}{\phi\beta}$, $\lambda_2 = \frac{1}{2}\frac{2\phi + \theta + \sqrt{\theta^2 - 4\theta^2\phi}}{\phi}$,and
$\lambda_3 = -\frac{1}{2}\frac{-2\phi - \theta + \sqrt{\theta^2 - 4\theta^2\phi}}{\phi}$
Since $\lambda_2 \not> 0$

Now for our last crtical point. We will use Routh-Hirwitz Theorem.

Theorem

Given the Polynomial,

$$P(\lambda) = \lambda^n + a_1 * \lambda^{n-1} + ... + a_{n-1} * \lambda + a_n$$

where the coefficients a_i *are real constants,* $i = 1, ..., n$*, define the n Huwitz matrices using the coefficients of* a_i *of the characteristic polynomial*

$$H_1 = \begin{pmatrix} a_1 \end{pmatrix}, H_2 = \begin{pmatrix} a_1 & 1 \\ a_3 & a_2 \end{pmatrix}, H_3 = \begin{pmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{pmatrix}$$

and

$$H_n = \begin{pmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & 1 & \cdots & 0 \\ a_5 & a_4 & a_3 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_n \end{pmatrix}$$

where $a_j = 0$ if $j > n$. All the roots of the polynomial $P(\lambda)$ are negative or have negative real parts iff the determinants of all Hurwitz matrices are positive:

$$detH_j > 0, j = 1, 2, ..., n.$$

When $n = 2$, the Routh-Hurwitz criteria simplify to $detH_1 = a_1 > 0$ and

$$detH_2 = det \begin{pmatrix} a_1 & 1 \\ 0 & a_2 \end{pmatrix} = a_1 a_2 > 0$$

or $a_1 > 0$ and $a_2 > 0$. For polynomials of degree $n = 2, 3, 4$ and 5, the Routh-Hurwitz criteria are as follows:

$$n = 2; a_1 > 0, a_2 > 0$$

$$n = 3; a_1 > 0, a_3 > 0, a_1 a_2 > 0$$

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Now we will analyze the critical point $(\frac{\beta\alpha}{\gamma}, \frac{\theta-\phi\beta\frac{\alpha}{\gamma}}{\eta}, \frac{\alpha}{\gamma})$. Plugging in our values for x, y, and z, we can attain our H_2 matrix. In this example we will keep our matrix in terms of x, y, and z instead of simplifying them at first.

$$J(\frac{\beta\alpha}{\gamma}, \frac{\theta-\phi\beta\frac{\alpha}{\gamma}}{\eta}, \frac{\alpha}{\gamma}) = \begin{pmatrix} -\beta z & 0 & \beta x \\ 0 & 0 & \gamma y \\ -\phi z & -\eta z & 0 \end{pmatrix}$$

To find the roots of the system, it is easiest to take the determinant of a $\lambda * I - H_2$, which yields:

$$\lambda * I - J(\frac{\beta\alpha}{\gamma}, \frac{\theta-\phi\beta\frac{\alpha}{\gamma}}{\eta}, \frac{\alpha}{\gamma}) = \begin{pmatrix} \lambda + \beta z & 0 & -\beta x \\ 0 & \lambda & -\gamma y \\ \phi z & \eta z & \lambda \end{pmatrix}$$

Which yeilds a $P(\lambda)$ that looks like this:

$$(\lambda + \beta z)(\lambda^2 + (\gamma\eta yz)) + (\lambda\beta\phi xz) = 0$$

Which can be simplified further to the form:

$$\lambda^3 + \lambda^2(\beta z) + \lambda(\gamma\eta yz + \beta\phi xz) + \beta\eta\gamma yz^2$$

Using Theorem 2, we can now look to show that $a_1 > 0$, $a_3 > 0$, and that $a_1 a_2 > a_3$.

It is clear that a_1 and a_3 are both non-negative regardless of the values of β, η, or γ, so all that is required is proving that $a_1 a_2 > a_3$.

$$a_1 a_2 = \beta\gamma\eta yz^2 + \beta^2\phi xz^2 > \eta\gamma\beta yz^2$$

A little bit of algebra will show that:

$$\beta^2\phi xz^2 > 0$$

Plugging in the values at our critical point gives us:

$$\frac{\beta^3\alpha^3\phi}{\gamma^3} > 0$$

Now By using Liapunov's Theorem we can prove global stability. Liapunov'sTheorem
Let (0,0) be an equailibrium of the autonomous system. And let V be a positive definite c^1 function in a nieghborhood U of the origin.

(i) $dV(x,y)/dt \leq 0 for$(x,y)$\in U-(0,0), then(0,0)isstable.$
(ii) $dV(x,y)/dt$ ¡ 0 for (x,y)
$\in U-(0,0), then(0,0)isasymtoticallystable.$
(iii) $dV(x,y)/dt$ ¿ 0 for (x,y)
$\in U-(0,0), then(0,0)isunstable.$