Exploring Minimal Time Paths through Isotropic Mediums

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Introduction

In our project, we investigated the least time paths through various combinations of isotropic mediums, using the example of a dog named Elvis playing fetch on a beach.

Least Time Paths

Using MATLAB to solve for and model the least time paths, we studied the following paths with equations T equal to the time it takes to travel from one point to another:

- **Shore to Sea** - from a point along the boundary of two mediums to a point in one of the mediums

$$T = \frac{a-x}{v_1} + \frac{\sqrt{x^2+b^2}}{v_2} \tag{1}$$

- **Land to Sea** - from a point in one medium to a point in another

$$T = \frac{\sqrt{x^2+a^2}}{v_1} + \frac{\sqrt{b^2+(d-x)^2}}{v_2} \tag{2}$$

- **Land to Shore to Sea** - from a point in one medium, then along the boundary between the two mediums, and then to a point in the other medium

$$T = \frac{\sqrt{{z_0}^2+{y_0}^2}}{v_1} + \frac{z_1-z_0}{v_2} + \frac{\sqrt{(x_1-z_1)^2+{y_1}^2}}{v_3} \tag{3}$$

- **Multiple Isotropic Mediums** - from a point in one medium, then through N mediums, and to a point in the $(N+1)th$ medium

$$T = \frac{\sqrt{(x_1-a_1)^2+(H-a_2)^2}}{v_1} + \sum_{i=2}^{n} \frac{\sqrt{(x_i-x_{i-1})^2+H^2}}{v_i} + \frac{\sqrt{(b_1-x_n)^2+(b_2-nH)^2}}{v_{n+1}} \tag{4}$$

- **Non-Horizontal Boundary between Two Mediums** - from a point in one medium to a point in another with a non-horizontal boundary separating the two mediums

$$T = \frac{\sqrt{(x-a_1)^2+(f(x)-a_2)^2}}{v_1} + \frac{\sqrt{(b_1-x)^2+(b_2-f(x))^2}}{v_2} \tag{5}$$

Modeling the Minimal Time Paths

By optimizing the time T for each case, we were able to find the least time paths analytically for (1) and numerically for (2), (3), (4), and (5). The following figures were produced in MATLAB in order to illustrate the least time paths of each situation.

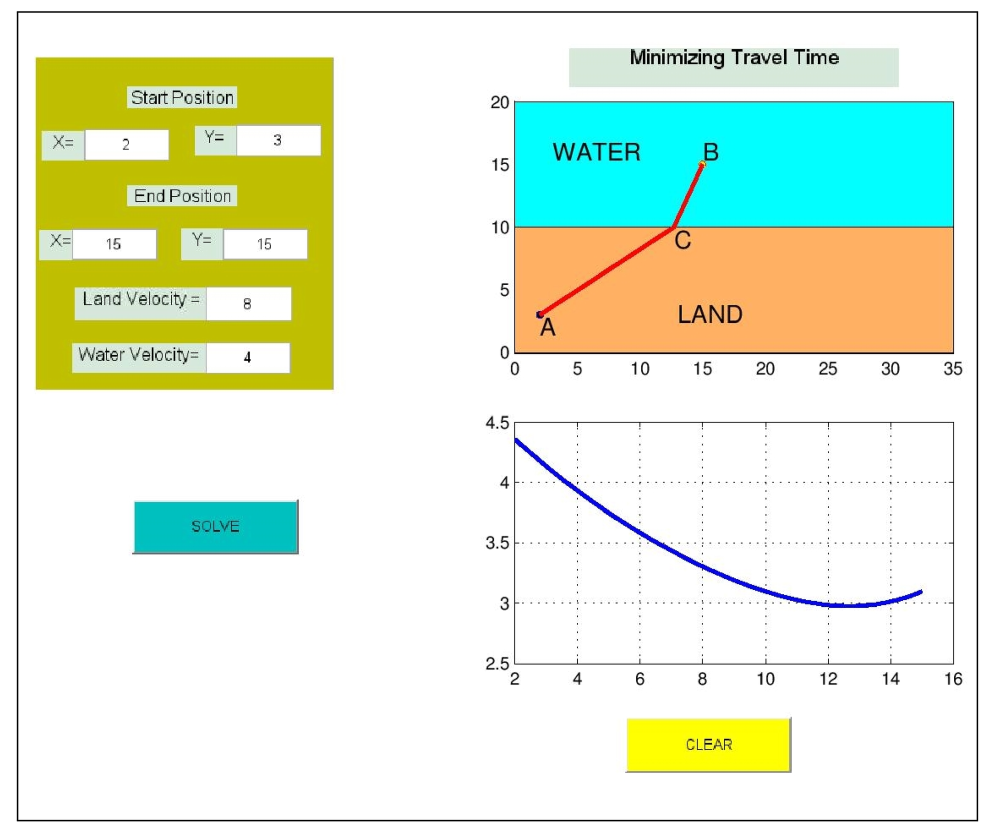

Figure 1: The Graphical User Interface (GUI) is designed to easily find a least time path when traveling from one isotropic medium to another. It allows a user to choose the coordinates of both the start and end point as well as the velocities on land and in water. By clicking the solve button, the GUI then yields two graphs. The top graph illustrates the minimal time path, while the bottom graph shows time as a function of x where x is the entry point into the water.

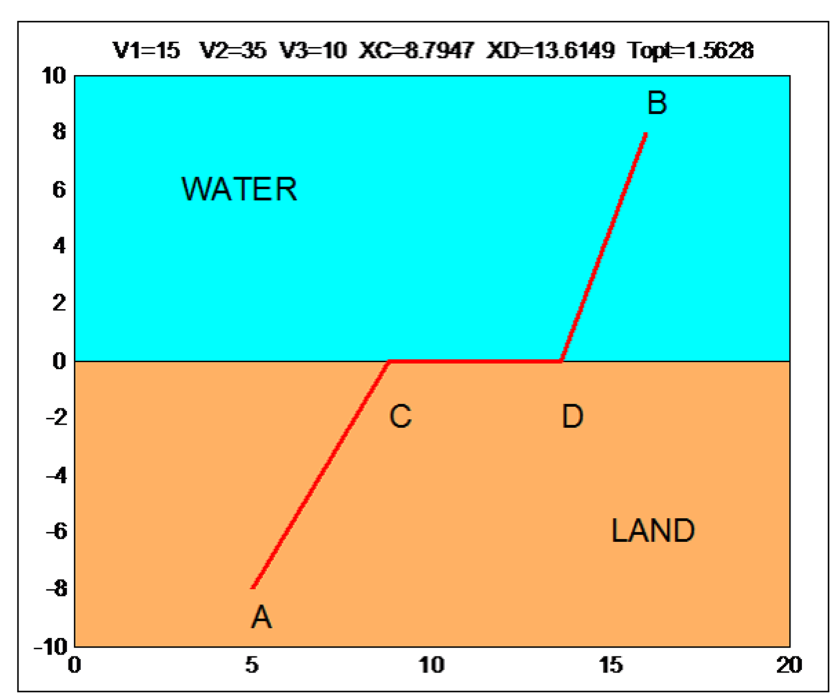

Figure 2: This figure models the least time path for the case when traveling from land, along shore, and to water. In this case, the velocity along the shore is greater than the velocity on land and the velocity in water.

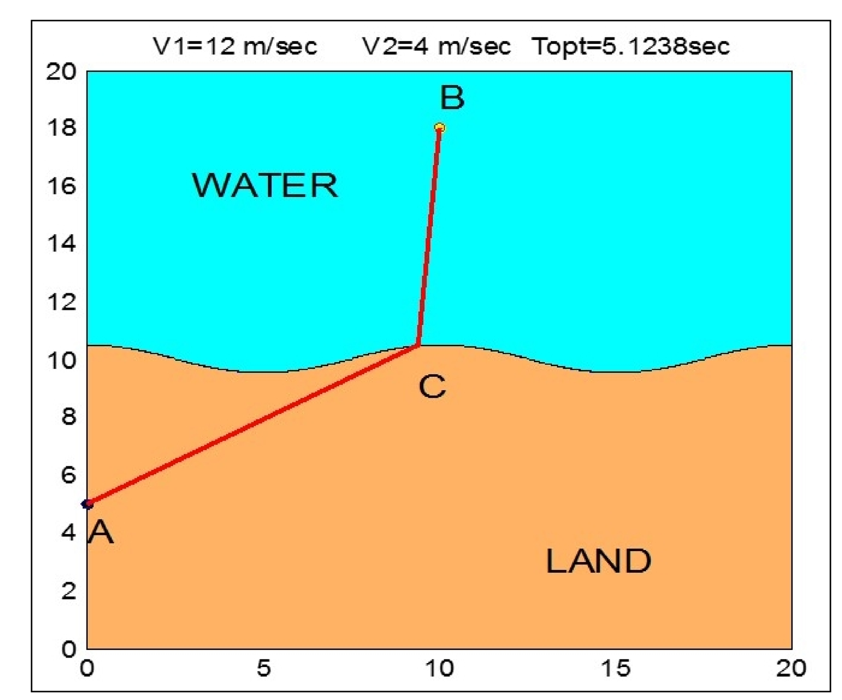

Figure 3: This figure models the least time path when traveling through two mediums with a boundary that can be written as the function $f(x) = \cos x$. In this case, the velocity on land is much faster than the velocity in water.

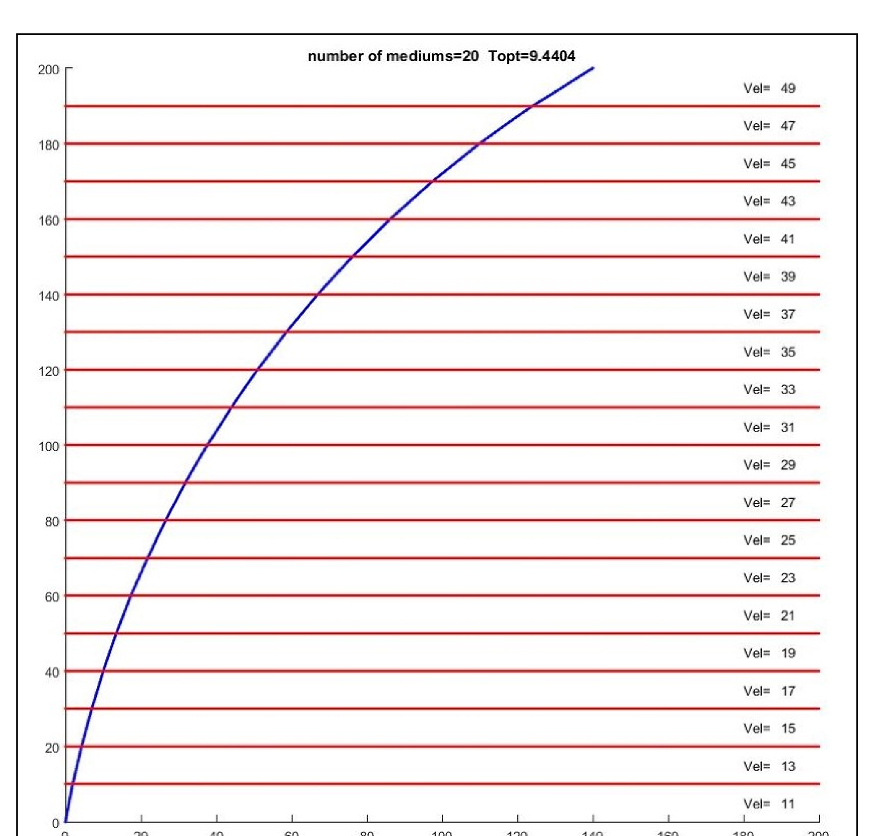

Figure 4: (left) This figure models the least time path when traveling through 20 mediums.

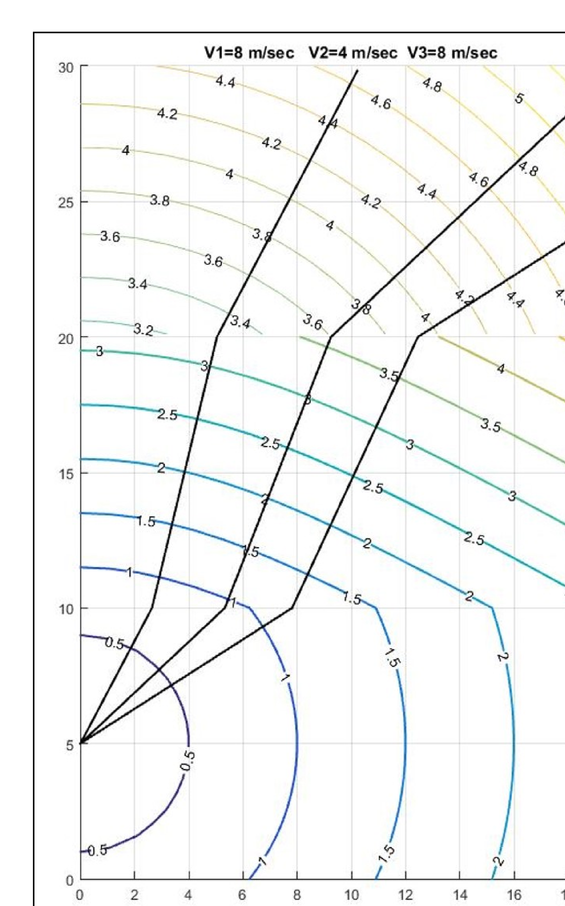

Figure 5: (right) This figure models the contour lines, or isochrones, and three potential least time paths when traveling through 3 mediums.

Applications

Some industrial applications include:

- **Navigation Software** - Navigation apps may account for traffic by turning the regular street map into a traffic density map, which changes the density of the traffic based on a real-time feed that takes into account the duration of traffic lights, weather, traffic, and various other factors. The app would use a model similar to our model for the multiple medium case in order to allow drivers to arrive at their destinations in the least amount of time.
- **Industrial Drilling and Tunneling** - The construction of tunnels often entails digging the shortest distance from point to point. However, our project is applicable to finding the most cost and time efficient passage for digging a tunnel because we can turn the map of the materials through which the tunnel passes into a map of mediums with different densities of "‘toughness to dig'' or "‘cost to dig.'' Again, this application uses a model similar to our multiple medium case.

Open Questions

Some open questions for potential further studies include finding the least time paths for the following cases:

- Multiple mediums with non-linear boundaries
- Multiple mediums with boundaries that can be traveled along
- Multidimensional mediums

Acknowledgments

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References

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- Pennings, T. (May, 2003). Do Dogs Know Calculus? *The College Mathematics Journal*, 34, 178-182.