Exploring Minimal Time Paths through Isotropic Mediums

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Introduction

Over the course of the semester, our main goal was to find the least time paths for various combinations of isotropic mediums. This problem originated from Timothy Pennings's paper "Do Dogs Know Calculus?" in which he discussed and modeled the behavior his dog, Elvis, playing fetch. Pennings used differential calculus to predict Elvis's optimized path to the ball so that he would reach a thrown ball in the least amount of time. In our project, we studied the minimal time paths of Elvis the dog from one point in one medium to another point in another medium, assuming a constant velocity in each medium.

Using MATLAB to solve for and model the minimal time paths, we explored a variety of cases. We started out with the simple case of traveling along a shoreline and then entering water. Building upon this, we then modeled the least time path from a point on land to a point in the water, where we were able to demonstrate that Snell's Law must hold for any least time path from one isotropic medium to another isotropic medium with a different constant velocity. Next, we calculated the least time path in which Elvis travels from a point on land, then along the shore, and finally to a point in the water. In each of these cases, we used MATLAB to find contour lines, or isochrones, that illustrate the possible locations that could be reached in a given amount of time. Lastly, we studied the least time path through a collection of any number of mediums as well as the case in which the path must travel between two mediums with a non-linear boundary.

Projects

Shore to Water

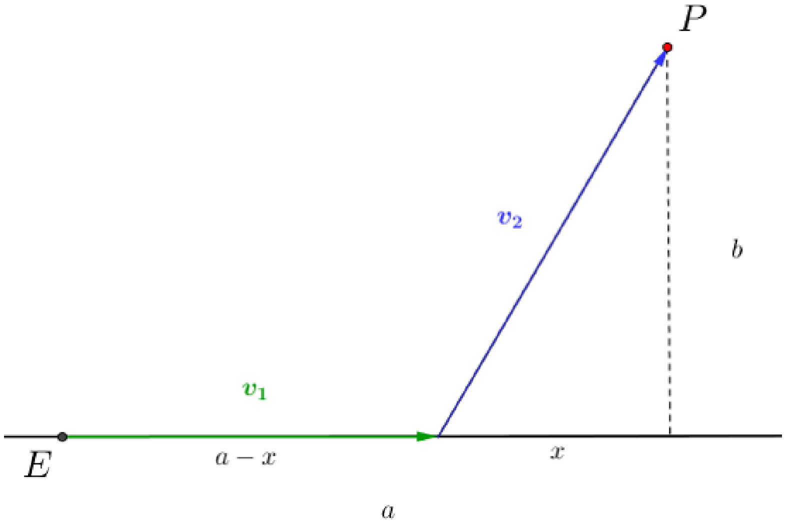

Figure 1: This diagram depicts the Shore to Water problem discussed below.

This case was especially nice because we were able to analytically solve for both the contours and the required path. We let v_1 = velocity along shore and v_2 = velocity in water. Then, the time it takes for Elvis to run from point E to point P is given by the equation:

$$T = \frac{a-x}{v_1} + \frac{\sqrt{x^2+b^2}}{v_2}$$

By optimizing the time T, we can find the distance x. This derivative of time T with respect to x is given by the equation:

$$\frac{dT}{dx} = -\frac{1}{v_1} + \frac{x}{v_2\sqrt{x^2+b^2}}$$

By setting $\frac{dT}{dx} = 0$, we find the distance x. The distance x is given by the equation:

$$x = \frac{bc}{\sqrt{1-c^2}} \text{ for } c = \frac{v_1}{v_2}.$$

Furthermore, we were able to analytically solve for the contour lines, or isochrones.

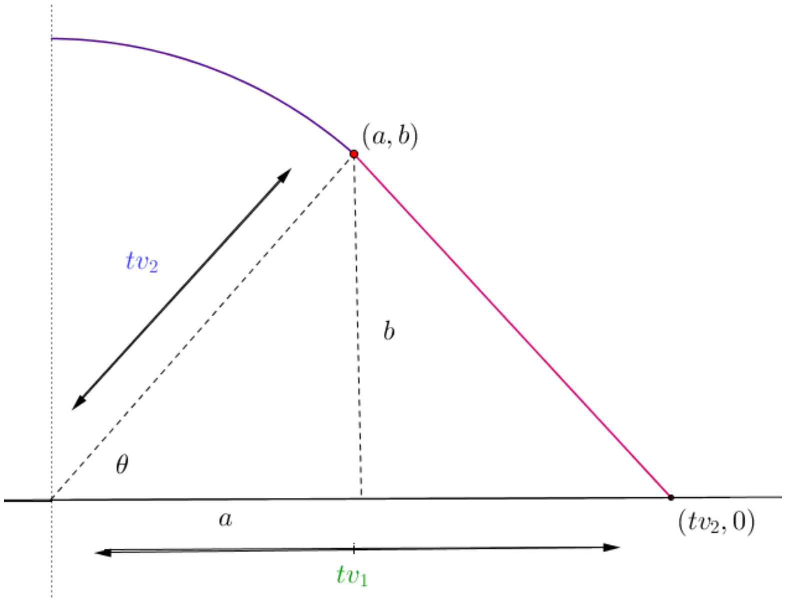

Figure 2: This diagram illustrates the contour lines discussed below.

For a given optimum time t, Elvis can travel to anywhere along the contour lines given by the following two equations:

- Case 1: $y = \frac{\sin\theta v_2}{v_1 - \cos\theta v_2}(x + tv_1)$, $x \in [tv_2 \cos\theta, tv_1]$
- Case 2: $y = \sqrt{t^2 {v_1}^2 - x^2}$, $x \in [0, tv_2 \cos\theta]$

Examples of the MATLAB figures that illustrate the contour lines can be found in the appendix.

Land to Water

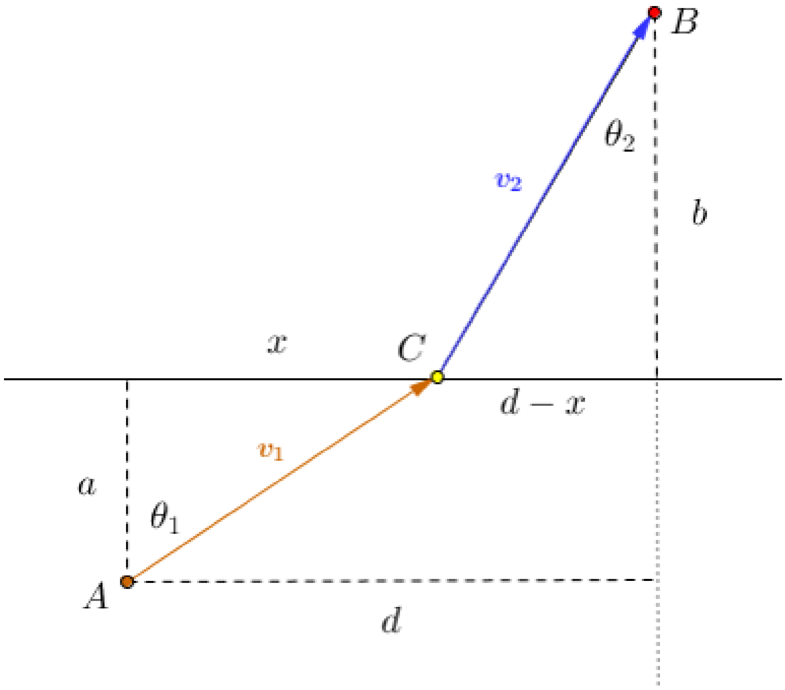

Figure 3: This diagram depicts the Land to Water problem discussed below.

Just as in the previous case, we modeled the time it takes for Elvis to run from point A on the land to point B in the water as a polynomial equation given by:

$$\begin{aligned} T &= \frac{|AC|}{v_1} + \frac{|BC|}{v_2} \\ &= \frac{\sqrt{x^2 + a^2}}{v_1} + \frac{\sqrt{b^2 + (d-x)^2}}{v_2} \end{aligned}$$

However, unlike the previous case, we found ourselves facing an equation of the variable x with no analytical solution, forcing us to find a numerical solution using MATLAB. This was derived by once again setting the derivative of time T with respect to x equal to 0. The equation to be solved numerically is given by:

$$\frac{dT}{dx} = \frac{x}{\sqrt{x^2 + a^2}} \frac{1}{v_1} + \frac{d-x}{\sqrt{(d-x)^2 + b^2}} \frac{1}{v_2}$$

By setting $\frac{dT}{dx} = 0$, we derive the following equation:

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$$

This formula is known as Snell's Law. This implies that Snell's Law is a condition that must be held when traveling along the least time path.

Furthermore, we were able to analytically solve for the contour lines, or isochrones.

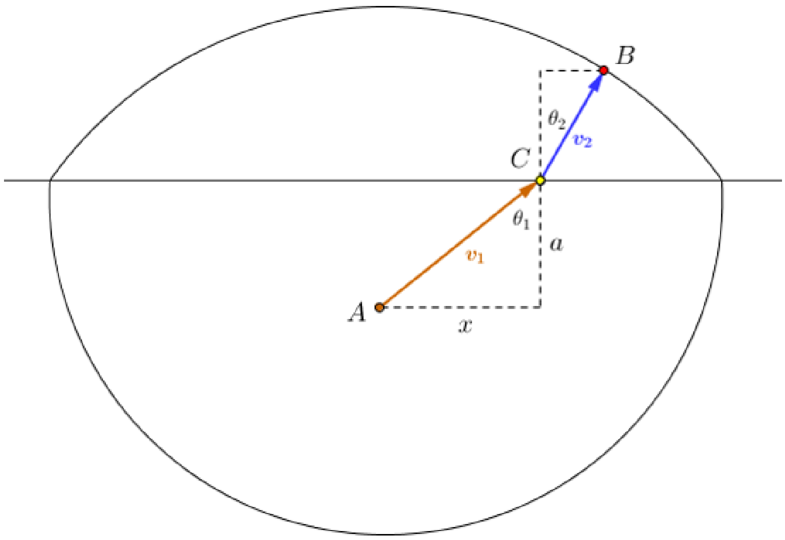

Figure 4: This diagram depicts the contour lines for the Land to Water problem as discussed below.

For a given a, x, T, v_1, and v_2, Elvis can travel to anywhere along the contour lines. Using Snell's Law, we found that

$$\frac{\sin\theta_1}{|v_1|} = \frac{\sin\theta_2}{|v_2|}$$

$$\frac{x|v_2|}{\sqrt{a^2+x^2}|v_1|} = \sin\theta_2$$

$$\theta_2 = \arcsin\left(\frac{x}{c\sqrt{a^2+x^2}}\right)$$

where $c = \frac{v_1}{v_2}$.

We also define the time travelled in water as $T_w = T - \frac{\sqrt{a^2+x^2}}{v_1}$. Therefore, we can find the countour lines, or more specifically the coordinates of the point $B = (\alpha, \beta)$, using the following equations:

- Case 1: If $0 < \sqrt{a^2 + x^2} < v_1 T$,

$$\begin{cases} \alpha = T_w v_2 \sin\theta_2 + x \\ \beta = T_w v_2 \cos\theta_2 \end{cases}$$

  or

$$\begin{cases} \alpha = (T - \frac{x}{c^2}) + x \\ \beta = (T - \frac{\sqrt{a^2+x^2}}{c}) \cos(\arcsin(\frac{x}{c\sqrt{a^2+x^2}})) \end{cases}$$

  where $c = \frac{v_1}{v_2}$.

- Case 2: If $\sqrt{a^2 + x^2} = v_1 T$,

$$\begin{cases} \alpha = x \\ \beta = \sqrt{{v_1}^2 T^2 - \alpha^2} \end{cases}$$

Using MATLAB, we also made a Graphical User Interface (GUI) for this case that will allow others to easily manipulate the variables in the equation for time T and view a model of the least time path accordingly. The GUI displays an upper graph that shows the least time path and a lower graph that shows time T as a function of x, where x is the entry point into the water. An image of the GUI can be found in the Appendix.

Land to Shore to Water

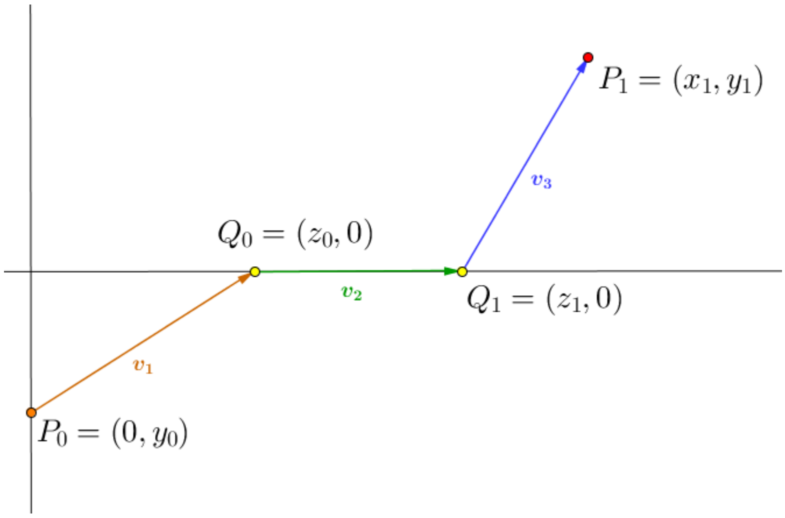

Figure 5: This diagram depicts the Land to Shore to Water problem discussed below.

In this case, we sought to find the least time path when there is the possibility of traveling along a boundary between two mediums, perhaps a shoreline or a bridge, where the boundary line has a higher velocity than either medium. In order to do this, we modeled the time as a function of the point Q_0 where the shoreline is entered and the point Q_1 where the shoreline is exited.

The time T it takes for Elvis to run from the point $P_0 = (0, y_0)$ on land, through the points $Q_0 = (z_0, 0)$ and $Q_1 = (z_1, 0)$ along the shore, and to the point $P_1 = (x_1, y_1)$ is given by the the equation:

$$T = \frac{\sqrt{{z_0}^2 + {y_0}^2}}{v_1} + \frac{z_1 - z_0}{v_2} + \frac{\sqrt{(x_1 - z_1)^2 + {y_1}^2}}{v_3}$$

We then took a multivariable derivative, or the gradient, which is given by the following equations:

$$\frac{\partial T}{\partial z_0} = \frac{z_0}{v_1\sqrt{{z_0}^2 + {y_0}^2}} - \frac{1}{v_2}$$

$$\frac{\partial T}{\partial z_1} = \frac{1}{v_2} - \frac{x_1 - z_1}{v_3\sqrt{(x_1 - z_1)^2 + {y_1}^2}}$$

We then used MATLAB to numerically solve for z_0 and z_1 by setting the gradient equal to 0. However, for this case, we leave the problem of finding the contour lines as an open question.

Multiple Isotropic Mediums

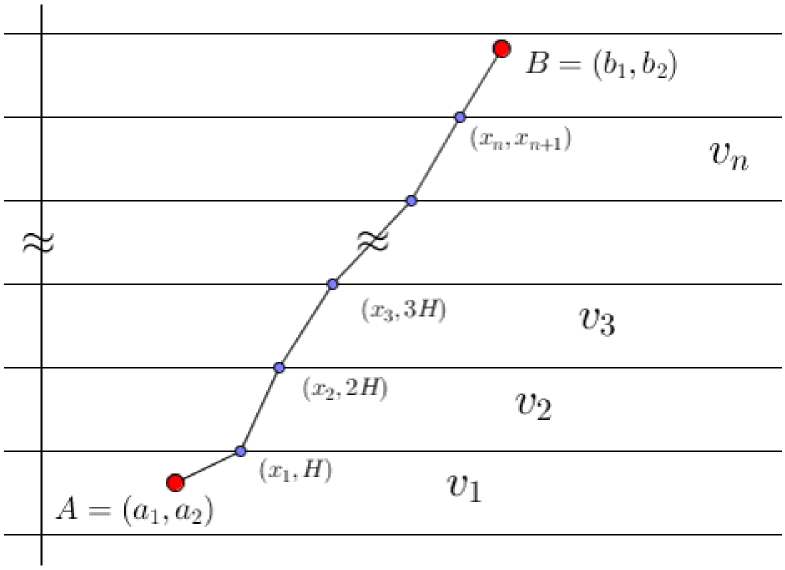

Figure 6: This diagram depicts the Multiple Mediums problem discussed below.

Finding the least time path through several mediums was a natural extension of the case of just two mediums. We were able to model time T as an equation of n variables, where there are $n+1$ mediums. The time function is given by the following:

$$T = \frac{\sqrt{(x_1 - a_1)^2 + (H - a_2)^2}}{v_1} + \sum_{i=2}^{n} \frac{\sqrt{(x_i - x_{i-1})^2 + H^2}}{v_i} + \frac{\sqrt{(b_1 - x_n)^2 + (b_2 - nH)^2}}{v_{n+1}}$$

Rather than explicitly computing a derivative, we used the `fminsearch` function in MATLAB on our time function to find a numerical solution.

Although this case provides a nice generalization of the two-medium case, it does not account for the possibility of traveling along the boundaries that separate the mediums, as is the case in the Land to Shore to Water problem. We leave this problem as another open question. Another limitation we faced was finding least time paths for a large number of mediums due to limitations of `fminsearch`. Finding this path is an important goal for future research as it could provide a model for the case a medium with a changing velocity.

For the contour plot, we were unable to find analytical solutions to the contours, so we found them numerically. We were also able to successfully use MATLAB to illustrate the contours for up to three mediums. Examples of this work can be found in the Appendix.

Non-Horizontal Boundaries between Two Mediums

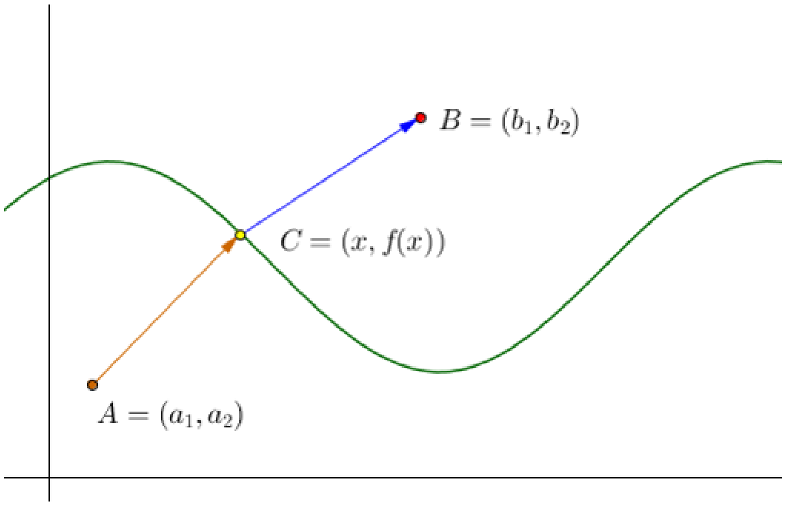

Figure 7: This diagram depicts the Non-Horizontal Boundary problem discussed below.

The last case we investigated consisted of finding the least time path between two isotropic mediums separated by a boundary that is not a horizontal line. We let $f(x)$ be the function that describes the boundary line and the point $(x, f(x))$ be any position along the boundary line. Applying this concept to the equation for time T in the Land to Water problem, we found that the time it takes to travel from point A on the land to point B in the water is given by the following equation:

Just as in the previous case, we modeled the time it takes for Elvis to run from point A on the land to point B in the water as a polynomial equation given by:

$$T = \frac{\sqrt{(x - a_1)^2 + (f(x) - a_2)^2}}{v_1} + \frac{\sqrt{(b_1 - x)^2 + (b_2 - f(x))^2}}{v_2}$$

The x position that allows for a minimum time can be found by the `fminsearch` function in matlab. We successfully modeled the cases where $f(x) = \cos x$ and $f(x) = mx + b$. For the latter, setting m equal to zero is another way to generate the boundary of the Land to Water problem. We have not yet found contours for this case, but they can surely be found with further research.

Conclusion

Applications

Navigation Software One application of our findings is in navigation software. Many navigation apps that currently exist give directions to a destination, but do not account for the traffic, which could possibly have a significant impact on total travel time. One way to account for traffic is to turn the regular street map into a traffic density map which changes the density of the traffic based on a real time feed that takes into account the duration of traffic lights, weather, traffic, and various other factors to create a model that is similar to what we did with the case of multiple mediums. This would allow commuters to arrive at their destinations faster. It would also allow businesses such as taxi, delivery, and logistics companies to be more time efficient.

Industrial Drilling and Tunneling Another application lies in industrial drilling and tunneling. The construction of tunnels often entails digging the shortest distance from point to point. A problem is that digging the shortest distance for underground or underwater tunnels is neither cost nor time efficient. For example, when digging a tunnel through a mountain, the mountain contains very different rocks or metals that might be encountered while digging, some of which are easier to drill through than others. Before beginning the construction of the tunnel, the construction team consults with geologists to map out the structure of the mountain. Based on the map and structure of the mountain the construction team decides the best tunneling passage. Our project is applicable to finding the best passage, because we can turn the map of mountain structure into a map of mediums with different densities of toughness to dig or cost to dig. This would be similar to our multiple medium case; however, instead of a time function, a cost function could be minimized and used to find least costly path to dig.

Open Questions

Several unanswered questions exist that we have not previously stated. First of all, it is a natural progression from the case of non-linear boundaries to find a least time path for multiple mediums with non-linear boundaries. Another extension of that case is to find the least time path where movement along the non-linear boundary is allowed, and the boundary has a higher velocity than the other mediums. For both the cases of linear and non-linear boundaries it would be useful to be able to find a least time path for a collection of multiple mediums where there are multiple boundaries that movement is allowed on. A case that we have not explored at all is allowing for multiple dimensions such as swimming under the water or flying in the air. It is unknown to us how this would change the overall structure of the problems.

Acknowledgements

We would like to especially thank Dr. Ameziane Harhad for sharing his knowledge by guiding us through this project and for his patience with us throughout the semester. We would also like to thank Dr. Peter Wolenski for his support and advise throughout the project as well.

References

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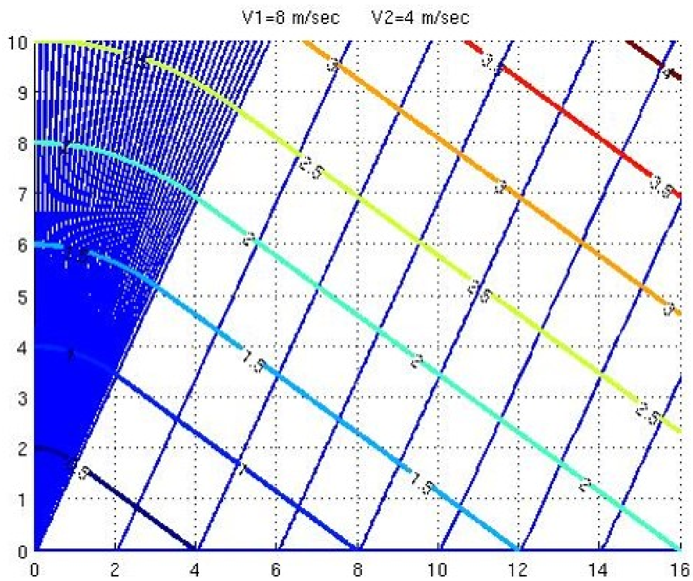
Jonsson, H. (Jan., 2010). *Path Optimization with Application to Tunneling.* Paper presented at Applied Parallel and Scientific Computing: 10^th International Conference, Reykjavik, Iceland. doi:10.1007/$978-3-642-28145-7_5$

Appendix

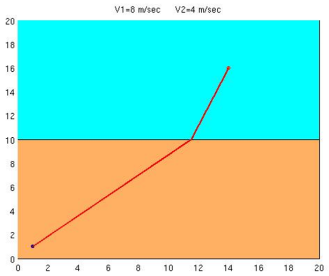
Shore to Water

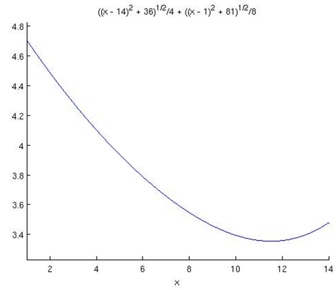
Contour Lines

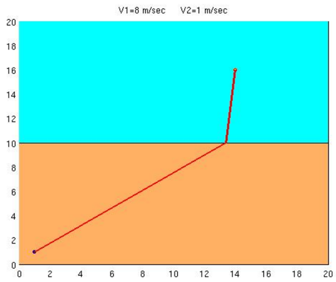


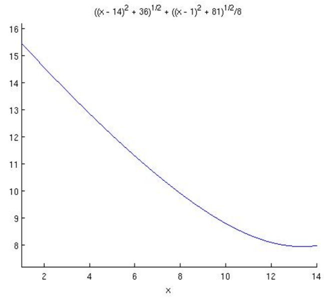
Land to Water

Least Time Path and Time Function

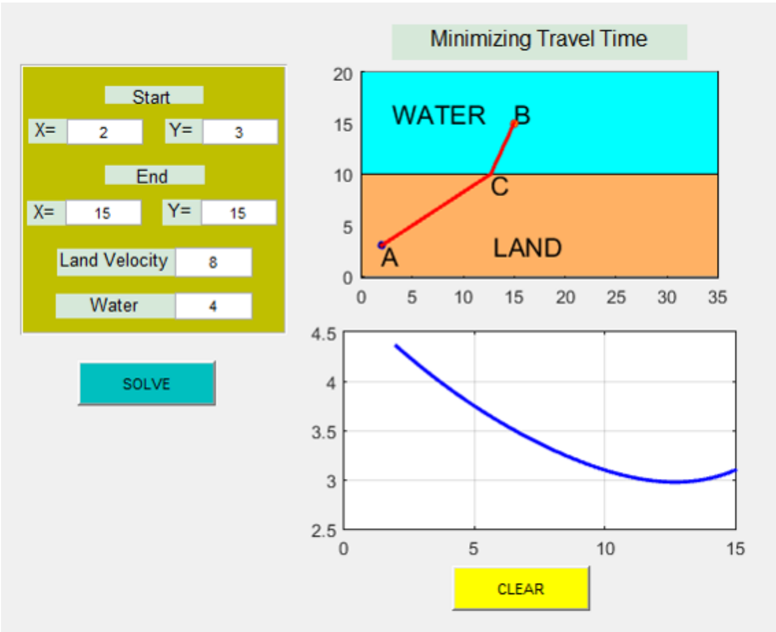






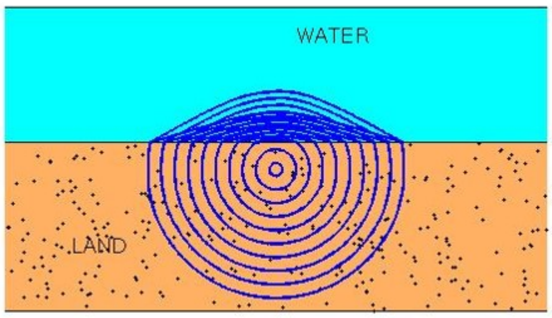


GUI - Miniziming Time

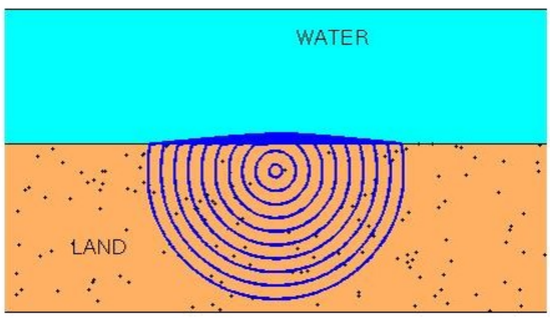


Contours

V1=8 m/sec   V2=4 m/sec

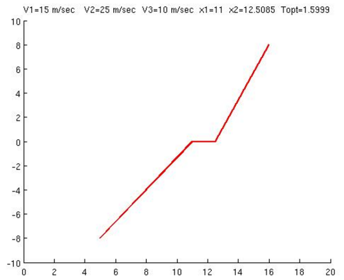


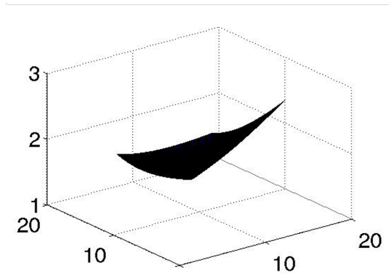
V1=100 m/sec   V2=10 m/sec

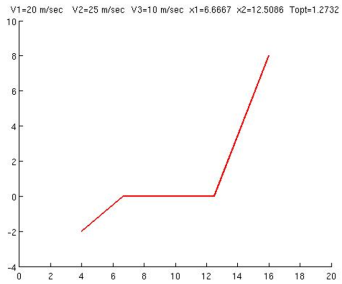


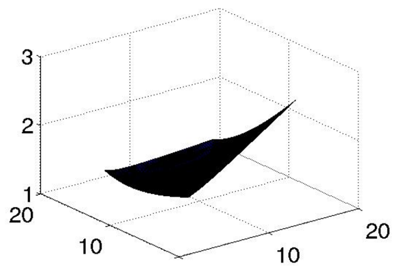
Land to Shore to Water

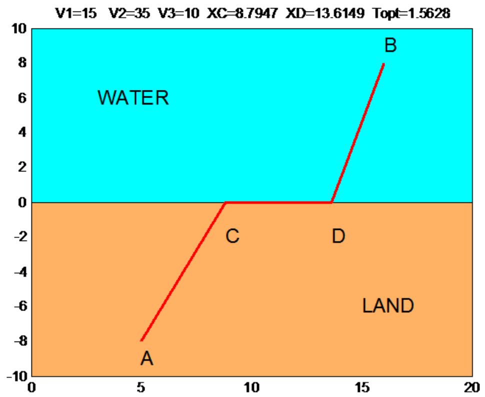
Least Time Path and Time Function





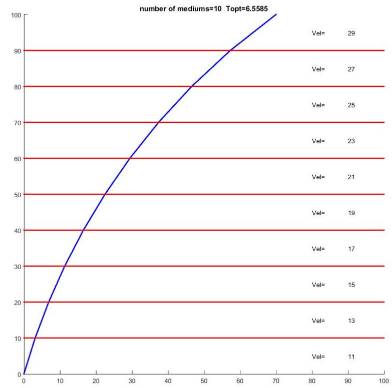


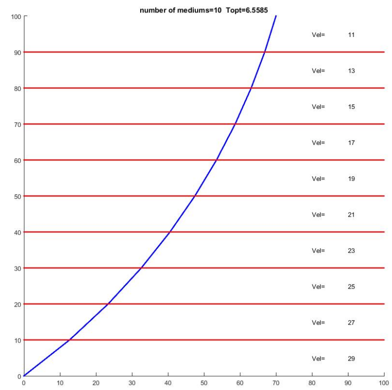


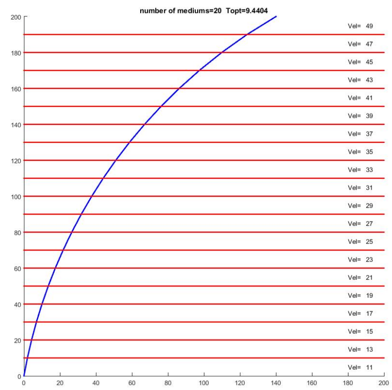


Multiple Mediums

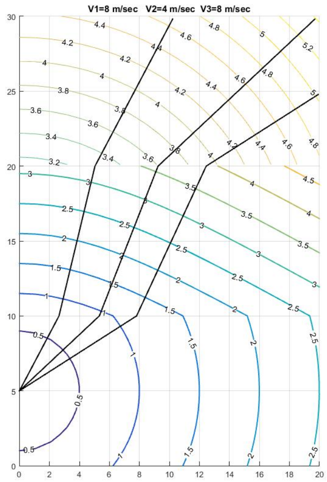
Least Time Path

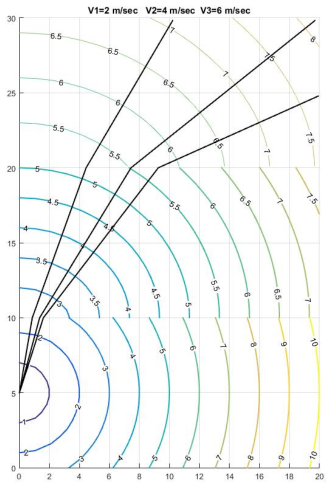






Contours





Non-Horizontal Boundary

Least Time Path