SOLVING NONLINEAR LEAST-SQUARES PROBLEMS WITH THE GAUSS-NEWTON AND LEVENBERG-MARQUARDT METHODS

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Abstract. We will analyze two methods of optimizing least-squares problems; the Gauss-Newton Method and the Levenberg Marquardt Algorithm. In order to compare the two methods, we will give an explanation of each method’s steps, as well as show examples of two different function types. The advantages and disadvantages will then be explored for both methods.

1. Introduction

1.1. Overview. An optimization problem begins with a set of independent variables and often includes conditions or restrictions that define acceptable values of the variables. Such restrictions are called restraints. The essential component is the objective function, which depends in some way on the variables. The solution of an optimization problem is a set of allowed values of the variables for which the objective function assumes an optimal value. In mathematical terms, optimization usually involves maximizing or minimizing; for example, maximizing profit or minimizing cost. In a large number of practical problems, the objective function $f(x)$ is a sum of squares of nonlinear functions

$$f(x) = \frac{1}{2} \sum_{j=1}^{m} (r_j(x))^2 = \frac{1}{2} ||r(x)||_2^2$$

that needs to be minimized. We consider the following problem

$$\min_x f(x) = \sum_{j=1}^{m} (r_j(x))^2.$$ 

This is a nonlinear least squares unconstrained minimization problem. It is called least squares because we are minimizing the sum of squares of these functions. Problems of this type occur when fitting model functions to data: if $\phi(x; t)$ represents the model function with $t$ as an independent variable, then each $r_j(x) = \phi(x; t_j) - y_j$, where $d(t_j, y_j)$ is
the given set of data points. Two common algorithms for solving such least-squares problems are the Gauss-Newton (GN) Method and the Levenberg-Marquardt Algorithm (LMA).

1.2. Terminology. The gradient $\nabla$ of a multivariable function $f$ is a vector consisting of the function’s partial derivatives:

$$\nabla f(x_1, x_2) = \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2} \right).$$

The Hessian matrix $H(f)$ of a function $f(x)$ is the square matrix of second-order partial derivatives of $r(x)$:

$$H(f(x_1, x_2)) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} \end{pmatrix}.$$

The transpose $A^\top$ of a matrix $A$ is the matrix created by reflecting $A$ over its main diagonal:

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}^\top = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Lastly, a matrix $A$ is positive-definite if, for all real non-zero vectors $z$, $z^\top Az > 0$. Equivalently, $A$ is positive-definite if every upper-left submatrix of $A$, including $A$ itself, has a positive determinant.

1.3. Newton’s Method. Newton’s method is an algorithm for locating roots that serves as the basis for the GN method. It is derived from the Taylor series expansion of a function $f(x)$ at a point $x = x_0 + \delta$:

$$f(x_0 + \delta) = f(x_0) + f'(x_0)\delta + \frac{1}{2}f''(x_0)\delta^2 + \ldots.$$

Newton’s method uses the first-order approximation

$$f(x_0 + \delta) \approx f(x_0) + f'(x_0)\delta,$$

which is the equation of the tangent line to the curve at an initial guess $x_0$. The point where this tangent intersects the $x$-axis will be the next guess $x_1$ and is given by

$$x_1 = x_0 + \delta_0 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$
Thus, Newton’s method is the iterative process
\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \]
Newton’s method can be used to approximate the roots of a differentiable function provided that the initial guess is reasonably close to the true root and the function’s derivative is not zero or very small in the neighborhood of the root. A slightly different version of Newton’s method can be used to find the extreme points of a function rather than its roots:
\[ x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)}. \]
This formulation of Newton’s method serves as the basis of the Gauss-Newton Method.

2. Least-Squares Problems

Least-Squares problems minimize the difference between a set of data and a model function that approximates this data. Given a set of data \( d(t_j, y_j) \) and a model function \( \phi(x; t_j) \), we obtain the difference of the functions with the equation \( r_j(x) = \phi(x; t_j) - y_j \), where \( y_j \) is the \( y \) component of the data point at \( t_j \). The objective function of least-squares problems is therefore
\[
(2.1) \quad f(x) = \frac{1}{2} \sum_{j=1}^{m} r_j^2(x).
\]
By minimizing \( f(x) \), we can find the parameters that most accurately match the model to the observed data. Each \( r_j \) is called a residual and is a smooth function from \( \mathbb{R}^n \) to \( \mathbb{R} \). This equation includes the sum of all components \( r_j \) of the residual vector \( r \) of \( m \) components given by the vector
\[
(2.1) \quad r(x) = (r_1(x), r_2(x), ..., r_m(x))^T.
\]
We can then rewrite (2.1) using the residual vector: \( f(x) = \frac{1}{2} \| r(x) \|_2^2 \).
When calculating the gradient of \( f(x) \), it is necessary to find the gradient of the residual vector. The Jacobian \( J(x) \) is a matrix of all \( \nabla r_j(x) \):
\[ J(x) = \begin{bmatrix} \frac{\partial r_j}{\partial x_i} \end{bmatrix}_{j=1,...,m; i=1,...,n} = \begin{bmatrix} \nabla r_1(x)^T \\ \nabla r_2(x)^T \\ \vdots \\ \nabla r_m(x)^T \end{bmatrix} \]

The gradient and Hessian of the objective can be expressed in terms of the Jacobian:

\begin{equation}
\nabla f(x) = \sum_{j=1}^{m} r_j(x) \nabla r_j(x) = J(x)^T r(x)
\end{equation}

\begin{equation}
\nabla^2 f(x) = \sum_{j=1}^{m} \nabla r_j(x) \nabla r_j(x)^T + \sum_{j=1}^{m} r_j(x) \nabla^2 r_j(x)
\end{equation}

\[
= J(x)^T J(x) + \sum_{j=1}^{m} r_j(x) \nabla^2 r_j(x)
\]

In many cases the gradient of each residual \( r_j \) is relatively easy to calculate, making the Jacobian a convenient substitution. \( J(x) \) and \( J(x)^T \) also comprise the first term of the Hessian matrix as seen in (2.3). The Hessian matrix must be positive definite for all least-squares problems. In cases where the residual is extremely close to the solution, \( \nabla^2 f(x) \) can be approximated by the first term, thus eliminating a rather lengthy calculation of \( \nabla^2 r(x) \) in the second term. This approximation is used for both the Gauss-Newton and Levenberg-Marquardt methods.

3. The Gauss-Newton Method

The Gauss-Newton method is based on the basic equation from Newton’s method (1.1), except that it uses a search direction vector \( p_k^{GN} \) and a step size \( \alpha_k \) in the revised equation

\begin{equation}
x_{k+1} = x_k + \alpha_k p_k.
\end{equation}

The values that are being altered in this case are the variables of the model function \( \phi(x; t_j) \). Like Newton’s method, GN is an iterative process, repeating equation (3.1) until the model function fits the data points satisfactorily. Using \( \nabla f(x) \) as \( f'(x) \) and the Hessian \( \nabla^2 f(x) \) as \( f''(x) \), this method generalizes Newton’s method for multiple dimensions. It also uses a different approximation of \( \nabla^2 f(x) \),

\[ \nabla^2 f(x) \approx J(x)^T J(x), \]
which eliminates the second term of the general equation (2.3) involving the sum of the residual Hessians $\nabla^2 r_j(x)$, for $j = 1, 2, ..., m$. This often saves a considerable amount of computational time. The gradient equation remains the same as the general equation involving $J(x)$ in (2.2). These approximations can be used to find the search direction $p_k^{GN}$ for each iteration with the equation

$$J_k^T p_k^{GN} = -J_k^T r_k.$$ (3.2)

Once we find $p_k^{GN}$ we must find the step length $\alpha_k$ for each iteration. We do this by minimizing $\phi(x_k + \alpha_k p_k^{GN})$; we minimize the model function $\phi(x)$ at the new point $x_k + \alpha_k p_k$ with respect to the variable $\alpha_k$ (where $\alpha_k > 0$). To find the optimal step length, we find $\alpha_k$ that satisfies the condition

$$\phi(x_k + \alpha_k p_k^{GN}) \leq \phi(x_k) + c_1 \alpha_k \nabla \phi(x_k)^T p_k^{GN}.$$ (3.3)

This method works for any function $\phi(x)$ that is bounded below, with any constant $c_1$ that satisfies the condition $0 < c_1 < 1$. This is an iterative process where we choose an initial guess for $\alpha_k$, plug it into (3.3), and alter it until the inequality is satisfied. There are other methods with more constraints that find even more optimal values of $\alpha_k$. These are explored in [4].

Gauss-Newton has the useful characteristic that when $J_k$ has full rank and the gradient $\nabla f_k$ is non-zero, $p_k^{GN}$ is a descent direction and thus a suitable direction for a line search. However, when $J_k^T p_k^{GN} = 0$, we know $J_k^T r_k = \nabla f_k = 0$, which indicates that $x_k$ is a stationary point and no further iterations are necessary. A restriction on this method is the requirement that $J_k$ must have full rank. GN is notable for its fast convergence close to the solution, but like Newton’s method, its efficiency depends on having an accurate initial guess.

4. Gradient Descent

Gradient descent, also known as steepest descent, is an optimization method which involves taking steps proportional to the negative gradient of the function at the current point. Its iteration scheme is

$$x_{k+1} = x_k - \lambda_k \nabla f(x_k).$$

This method will not be discussed in detail here, but it is worth noting that it performs in a manner opposite to that of Gauss-Newton: gradient descent will quickly approach the solution from a distance, but its convergence will become very slow when it is close to the solution.
5. The Levenberg-Marquardt Method

Another method commonly used to minimize least-squares problems is the Levenberg-Marquardt method. This method uses the same approximation for the Hessian matrix as Gauss-Newton but implements a trust region strategy instead of a line search technique. At each iteration we must minimize $p_k$ in the equation

\[
\frac{1}{2}||J_k p_k + r_k||^2, \quad \text{subject to } ||p_k|| \leq \Delta_k,
\]

where $\Delta_k > 0$ is the trust region radius, making a spherical trust region. The value for $\Delta_k$ is chosen for each iteration depending on the error value of the corresponding $p_k$. Given a step $p_k$ we then consider the ratio

\[
\rho_k = \frac{d(x_k) - d(x_k + p_k)}{\phi_k(0) - \phi_k(p_k)}
\]

which is a comparison of the actual reduction in the numerator and the predicted reduction in the denominator. The actual reduction gives the difference of values in the set of data $d(x)$ at points $x_k$ and $x_k + p_k$, while the predicted reduction gives the difference of values in the model function $\phi_k$ at $x_k$ (where the direction vector is 0) and at $x_k + p_k$ (after taking step $p_k$). If $\rho_k$ is close to 1, the model is an accurate approximation of the data and it is safe to expand the trust-region radius $\Delta_k$. If $\rho_k$ is positive but significantly smaller than 1 (meaning both the model and the objective functions are decreasing, but the model function is decreasing faster than the objective function), we can keep our value of $\Delta_k$. If $\rho_k$ is close to zero or negative, we must reduce $\Delta_k$ at the next iteration. Once a good $\Delta_k$ is found we can solve for $p_k$.

When the search direction $p_k^{GN}$ of the GN method lies inside this trust region (when $||p_k^{GN}|| \leq \Delta_k$), then $p_k^{GN}$ can be used to minimize (5.1). If $p_k^{GN}$ does not lie inside the trust region, we use the following equation to find a new search direction vector $p_k^{LM}$ that lies inside the trust region ($||p_k^{LM}|| = \Delta_k$).

\[
(J_k^T J_k + \lambda I)p_k^{LM} = -J_k^T r_k, \quad \text{where } \lambda > 0.
\]

In order to find an appropriate $\lambda$ for this equation, we reorder the variables to make it a function of $\lambda$:

\[
p_k(\lambda) = -(J_k^T J_k + \lambda I)^{-1}J_k^T r_k
\]
Since we are trying to find a value of $p_k$ that satisfies the equality $||p_k^{LM}|| = \Delta_k$, we want to find the norm of both sides of the equation and set it equal to our established value of $\Delta_k$. This is a complicated process that involves finding a new equation of diagonal and orthogonal matrices that are equivalent to $\nabla^2 f(x)$ and $\nabla f(x)$ and whose norms can be easily evaluated. For more information of this method refer to [4]. For simplification, we will assume a value for $\lambda$ and alter it based on error of previous iterations. Typically $\lambda_1$ is chosen to be small. If after the first iteration of (5.2) $||p_k|| > \Delta$, then $\lambda_2$ is chosen to be less than $\lambda_1$. Likewise, if $||p_k|| < \Delta$, then $\lambda_2$ is chosen to be greater than $\lambda_1$. After finding $p_k^{LM}$ we will substitute it into (3.1) and update our values for the variables of the model function. We do not need to worry about the step size $\alpha_k$, since $||p_k||$ is already established by the trust region.

LMA is a superior method to GN when $f(x)$ is large, because we account for the omitted value of the Hessian with $\lambda$. In effect, LMA uses a gradient-descent method when $f(x)$ is large, and switches to an altered GN method implementing a trust-search region when $f(x)$ is small. This trust-search region technique is more accurate, but often takes more iterations when it is far away from the minimum. LMA is also especially useful when $J_k$ is rank deficient, in which case GN would not be effective.

6. Example: Exponential Data

Here is a set of data for the United States population (in millions) and the corresponding year [5]:

<table>
<thead>
<tr>
<th>Year</th>
<th>Population</th>
</tr>
</thead>
<tbody>
<tr>
<td>1815</td>
<td>8.3</td>
</tr>
<tr>
<td>1825</td>
<td>11.0</td>
</tr>
<tr>
<td>1835</td>
<td>14.7</td>
</tr>
<tr>
<td>1845</td>
<td>19.7</td>
</tr>
<tr>
<td>1855</td>
<td>26.7</td>
</tr>
<tr>
<td>1865</td>
<td>35.2</td>
</tr>
<tr>
<td>1875</td>
<td>44.4</td>
</tr>
<tr>
<td>1885</td>
<td>55.9</td>
</tr>
</tbody>
</table>

This is the set of data points $d(t_j, y_j)$ where $t_j$ is the year and $y_j$ is the population. For the sake of convenience, year 1815 will be labelled as 1, 1825 as 2, etc. The model function is $\phi(x; t) = x_1 e^{x_2 t}$. Here is a
graph of the data points and the model with $x_1 = 6$ and $x_2 = .3$ as our initial guesses:

![Graph](image)

**Figure 1.** The exponential data and first approximation of the model function.

Our residual vector is a vector of the components $r_j = 6e^{3x} - y_j$.

$$r(x) = \begin{pmatrix}
6e^{3(1)} - 8.3 \\
6e^{3(2)} - 11 \\
6e^{3(3)} - 14.7 \\
6e^{3(4)} - 19.7 \\
6e^{3(5)} - 26.7 \\
6e^{3(6)} - 35.2 \\
6e^{3(7)} - 44.4 \\
6e^{3(8)} - 55.9
\end{pmatrix} = \begin{pmatrix}
-0.200847 \\
-0.0672872 \\
0.0576187 \\
0.220702 \\
0.190134 \\
1.09788 \\
4.59702 \\
10.2391
\end{pmatrix}$$

The goal is to minimize the least-squares problem, $f(x) = \frac{1}{2}||r(x)||_2^2$. We find $||r(x)||_2^2$ by adding the squares of all the entries in $r(x)$. Our result is $||r(x)||_2^2 = 127.309$. Therefore $f(x) = 63.6545$.

The Jacobian is necessary for both the Gauss-Newton and Levenberg-Marquardt methods. Recall that the Jacobian is a matrix of all the components of $\nabla r(x)$. We compute each $\nabla r_j(x)$ by finding the gradient with respect to $x_1$ and $x_2$ and then substituting the original values for $x_1$ and $x_2$. 
Now find the search direction vector $p_k$ using (3.2) for the GN method. For this first iteration $p_1 = (0.923529, -0.0368979)$. The approximations of $x_1$ and $x_2$ can now be updated:

$$x_{11} = 6 + 0.923529 = 6.92353$$
$$x_{21} = .3 - 0.0368979 = 0.263103$$

We repeat the GN method using these new approximations. At the third iteration,

$$x_{13} = 7.00009$$
$$x_{23} = 0.262078,$$

which yields the graph:

![Graph](image-url)

**Figure 2.** The exponential data and model function after the third iteration of the GN method.

At the third iteration we also have the residual vector
\[
\begin{pmatrix}
0.797515 \\
0.823383 \\
0.665998 \\
0.270077 \\
-0.746333 \\
-1.46989 \\
-0.563416 \\
1.07124 \\
\end{pmatrix}
\]

so that \(||r||^2 = 6.01308\) and \(f(x) = 3.00654\), indicating that the model has become quite accurate.

The Levenberg-Marquardt method can also be used to minimize this least squares problem. First, we must establish a value for \(\Delta\). For simplicity, we will assume a small value for the trust-region radius, \(\Delta = 0.05\). Since \(||p_{\text{GN}}^1|| = 0.924266 > \Delta\) we need to use (5.1) to find an appropriate \(p_{\text{LM}}^1\). Since the calculations for \(||r||\) are the same as GN, we use (2.1) to calculate the objective function: \(f(x) = 63.65305\). Because this number is relatively large, it is necessary to use equation (5.2). For the first iteration we substitute \(\lambda = 1\) and use the same approximations for \(J(x)\) and \(r\) to obtain the solution \(p_{\text{LM}}^1 = (0.851068, -0.0352124)\).

The approximations of \(x_1\) and \(x_2\) can now be updated and used to find the new value of \(f(x)\):

\[
\begin{align*}
x_{1_1} &= 6 + 0.851068 = 6.85107 \\
x_{2_1} &= 0.3 - 0.0352124 = 0.264788 \\
f(x) &= 3.08479 \\
\end{align*}
\]

Since \(||p_{\text{LM}}^1|| = 0.851796 > \Delta\) and the error has decreased, we decrease the value for \(\lambda\) to 0.1 and repeat this method using these new approximations. At the second iteration we get \(||p_{\text{LM}}^2|| = 0.150782 > \Delta\). At the third, \(||p_{\text{LM}}^3|| = 0.000401997 < \Delta\) and

\[
\begin{align*}
x_{1_3} &= 7.00005 \\
x_{2_3} &= 0.262079 \\
f(x) &= 3.00654 \\
\end{align*}
\]

The graph of the model function and data points at this iteration is shown below.

If we were to continue this LMA process, we would no longer use (5.2) because our new \(p_{\text{LM}}^3\) is inside the trust region. Instead we would minimize each following \(p_k\) in (5.1).
Now the results of these two methods can be compared to see which was more effective in this specific example. At the third iteration of each method $f(x) = 3.00824$ is the result for GN, and $f(x) = 3.00796$ is the result for LMA, demonstrating that LMA was more accurate in this example. The difference between these two methods is almost negligible in this case. Had the Jacobian of this function been rank-deficient, LMA would have been required.

7. Example: Sinusoidal Data

Here is a list of average monthly high temperatures for the city of Baton Rouge, LA[1]:

<table>
<thead>
<tr>
<th>Month</th>
<th>Temperature</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jan</td>
<td>61</td>
</tr>
<tr>
<td>Feb</td>
<td>65</td>
</tr>
<tr>
<td>Mar</td>
<td>72</td>
</tr>
<tr>
<td>Apr</td>
<td>78</td>
</tr>
<tr>
<td>May</td>
<td>85</td>
</tr>
<tr>
<td>Jun</td>
<td>90</td>
</tr>
<tr>
<td>Jul</td>
<td>92</td>
</tr>
<tr>
<td>Aug</td>
<td>92</td>
</tr>
<tr>
<td>Sep</td>
<td>88</td>
</tr>
<tr>
<td>Oct</td>
<td>81</td>
</tr>
<tr>
<td>Nov</td>
<td>72</td>
</tr>
<tr>
<td>Dec</td>
<td>63</td>
</tr>
</tbody>
</table>

This is the set of data points $d(t_j, y_j)$ where $t_j$ is the month and $y_j$ is the temperature. January will be labelled as 1, February as 2, etc. The model function is $\phi(x; t) = x_1 \sin(x_2 t + x_3) + x_4$. Here is a graph of the data points and the model with the initial guesses $(x_1, x_2, x_3, x_4) = (17, 0.5, 10.5, 77)$:
The residual vector:

\[
\begin{pmatrix}
17 \sin(.5(1) + 10.5) + 77 - 61 \\
17 \sin(.5(2) + 10.5) + 77 - 65 \\
17 \sin(.5(3) + 10.5) + 77 - 72 \\
17 \sin(.5(4) + 10.5) + 77 - 78 \\
17 \sin(.5(5) + 10.5) + 77 - 85 \\
17 \sin(.5(6) + 10.5) + 77 - 90 \\
17 \sin(.5(7) + 10.5) + 77 - 92 \\
17 \sin(.5(8) + 10.5) + 77 - 92 \\
17 \sin(.5(9) + 10.5) + 77 - 88 \\
17 \sin(.5(10) + 10.5) + 77 - 81 \\
17 \sin(.5(11) + 10.5) + 77 - 72 \\
17 \sin(.5(12) + 10.5) + 77 - 63
\end{pmatrix}
= \begin{pmatrix}
-0.999834 \\
-2.88269 \\
-4.12174 \\
-2.12747 \\
-0.85716 \\
0.664335 \\
1.84033 \\
0.893216 \\
0.0548933 \\
-0.490053 \\
0.105644 \\
1.89965
\end{pmatrix}
\]

\[||r||^2 = 40.0481\]
At the second iteration, (3.2). The first iteration yields

\[
J(x) = \begin{pmatrix}
\sin(x_2 + x_3) & x_1 \cos(x_2 + x_3) & x_1 \cos(x_2 + x_3) \\
\sin(2x_2 + x_3) & 2x_1 \cos(2x_2 + x_3) & x_1 \cos(2x_2 + x_3) \\
\sin(3x_2 + x_3) & 3x_1 \cos(3x_2 + x_3) & x_1 \cos(3x_2 + x_3) \\
\sin(4x_2 + x_3) & 4x_1 \cos(4x_2 + x_3) & x_1 \cos(4x_2 + x_3) \\
\sin(5x_2 + x_3) & 5x_1 \cos(5x_2 + x_3) & x_1 \cos(5x_2 + x_3) \\
\sin(6x_2 + x_3) & 6x_1 \cos(6x_2 + x_3) & x_1 \cos(6x_2 + x_3) \\
\sin(7x_2 + x_3) & 7x_1 \cos(7x_2 + x_3) & x_1 \cos(7x_2 + x_3) \\
\sin(8x_2 + x_3) & 8x_1 \cos(8x_2 + x_3) & x_1 \cos(8x_2 + x_3) \\
\sin(9x_2 + x_3) & 9x_1 \cos(9x_2 + x_3) & x_1 \cos(9x_2 + x_3) \\
\sin(10x_2 + x_3) & 10x_1 \cos(10x_2 + x_3) & x_1 \cos(10x_2 + x_3) \\
\sin(11x_2 + x_3) & 11x_1 \cos(11x_2 + x_3) & x_1 \cos(11x_2 + x_3) \\
\sin(12x_2 + x_3) & 12x_1 \cos(12x_2 + x_3) & x_1 \cos(12x_2 + x_3)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
-0.99999 & 0.0752369 & 0.0752369 \\
-0.875452 & 16.4324 & 8.21618 \\
-0.536573 & 43.0366 & 14.3455 \\
-0.0663219 & 67.8503 & 16.9626 \\
0.420167 & 77.133 & 15.4266 \\
0.803784 & 60.6819 & 10.1137 \\
0.990607 & 16.2717 & 2.32453 \\
0.934895 & -48.2697 & -6.03371 \\
0.650288 & -116.232 & -12.9147 \\
0.206467 & -166.337 & -16.6337 \\
-0.287903 & -179.082 & -16.2802 \\
-0.711785 & -143.289 & -11.9407
\end{pmatrix}
\]

As before, we will use GN and find the direction vector \( p_k \) using (3.2). The first iteration yields \( p_1 = (-0.904686, -0.021006, 0.230013, -0.17933) \), and so \( x_1 \) through \( x_4 \) can be updated:

\[
x_{11} = 17 - 0.904686 = 16.0953 \\
x_{21} = 5 - 0.021006 = 4.978994 \\
x_{31} = 10.5 + 0.230013 = 10.73 \\
x_{41} = 77 - 0.17933 = 76.8207
\]

At the second iteration,

\[
x_{12} = 17.5078 \\
x_{22} = 0.463583 \\
x_{32} = 10.8584 \\
x_{42} = 76.1255
\]
and \( ||r||^2 = 13.6556 \).

![Figure 5](image-url)

**Figure 5.** The sinusoidal data and model function after the second iteration of GN.

Since \( ||r|| = 6.32836 \) and \( ||p_{1}^{GN}|| = 0.95077 \) initially, the LMA can also be used. Again starting with \( \lambda = 1 \), we find that \( p_1 = (-0.7595, -0.0219004, 0.236647, -0.198876) \) and

\[
\begin{align*}
x_{11} &= 17 - 0.7595 = 16.2405 \\
x_{21} &= 0.5 - 0.0219004 = 0.4781 \\
x_{31} &= 10.5 + 0.236647 = 10.7366 \\
x_{41} &= 77 - 0.198876 = 76.8011
\end{align*}
\]

Now \( ||r||^2 = 13.6458 \) and \( ||p_{1}^{LM}|| = 0.820289 \), so \( \lambda \) is decreased by a factor of 10 and the process continues. At the second iteration, \( ||p_{2}^{LM}|| = 0.566443 \),

\[
\begin{align*}
x_{12} &= 16.5319 \\
x_{22} &= 0.465955 \\
x_{32} &= 10.8305 \\
x_{42} &= 76.3247
\end{align*}
\]

and \( ||r||^2 = 13.0578 \). Since \( ||r||^2 = 13.6556 \) after two iterations of GN, LMA is slightly more efficient in this example. The process could be continued for \( \Delta < ||p_{2}^{LM}|| = 0.566443 \).
8. Other Methods

In cases where the residual is large and the model does not fit the function well, methods other than Gauss-Newton and Levenberg-Marquardt must be used. This is because both GN and LMA approximate $\nabla^2 f(x)$ by eliminating the second term of (2.3) involving the calculation of $\nabla^2 r(x)$. Some of these methods are explored in [2].

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