APPLICATIONS OF THE UMBRAL CALCULUS

Abstract. The umbral calculus formalized by Roman and Rota has proven to be a fruitful mathematical method. Here we examine the sequence of telephone numbers and the sequence of Hermite polynomials, applying umbral methods to each. In particular, we offer a detailed proof of an interesting theorem by Gessel regarding the Hermite polynomials.

1. Umbrae

Given a sequence a_n, the *exponential generating function* (EGF) of a_n is

$$A(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n.$$

If $a_n = \alpha^n$ for some real number α, then

$$A(x) = \sum_{n=0}^{\infty} \frac{(\alpha x)^n}{n!} = e^{\alpha x}.$$

Many formulas in combinatorics become easy to derive using EGFs if we substitute a_n to A^n for a variable A, manipulate the EGF, then substitute A^n back to a_n. The variable A is called the *umbra,* which is Latin for shadow. This process involves the application of a linear functional L whose domain is the vector space of polynomials in A with real coefficients and whose codomain is the real numbers. Hence, $L(A^n) = a_n$ Because L is linear, we have for any polynomials in A p and q and for any real-valued function t,

$$L(p(A) + q(A)) = L(p(A)) + L(q(A)), \text{ and}$$
$$L(tp(A)) = tL(p(A))$$

2. Hermite Polynomials

The Hermite polynomials are a sequence of polynomials with applications in combinatorics, probability, and physics. They were first studied by Laplace in 1810 and then by Chebyshev in 1859. They were named after the French mathematician Charles Hermite, who made a substantial contribution to their study in 1864. Two types of Hermite polynomials exist, the probabilists' Hermite polynomials, $He_n(u)$, and the physicists' Hermite polynomials, $H_n(u)$, but each are merely a rescaling of the other according to $H_n(u) = 2^{n/2} He_n(u\sqrt{2})$. In this paper, we refer solely to the physicists' Hermite polynomials $H_n(u)$, which can be defined using an umbra we shall call M. For any positive integer n, define the umbra M to be

$$L(M^{2n+1}) = 0,$$
$$L(M^{2n}) = \frac{(-1)^n(2n)!}{n!}.$$

Then the Hermite polynomials, $H_n(u)$ are defined by

$$H_n(u) = L\big((M + 2u)^n\big).$$

Example 2.1. Here we show the derivation of the first three Hermite polynomials, H_0, H_1, and H_2 using the given umbral definition:

$$H_0(u) = L\big((M + 2u)^0\big) = L(M^0) = 1.$$

$$\begin{aligned} H_1(u) &= L(M + 2u) \\ &= L(M^1) + 2u \\ &= 0 + 2u \\ &= 2u. \end{aligned}$$

$$\begin{aligned} H_2(u) &= L\big((M + 2u)^2\big) \\ &= L(M^2 + 4uM + 4u^2) \\ &= L(M^2) + 4uL(M^1) + 4u^2 \\ &= -2 + 0 + 4u^2 \\ &= 4u^2 - 2. \end{aligned}$$

Next we wish to find an expression for the EGF of $H_n(u)$, denoted $F(u, x)$. The following lemmas are useful in finding this expression. The proof for Lemma 2.3 is given by DeAngelis [2].

Lemma 2.2. *For any positive integer α,*

$$\sum_{n=0}^{\infty} H_{\alpha n}(u)\,\frac{x^n}{n!} = L\big(e^{(M+2u)^\alpha x}\big)$$

Proof.

$$\begin{aligned}
\sum_{n=0}^{\infty} H_{\alpha n}(u)\,\frac{x^n}{n!} &= \sum_{n=0}^{\infty} L\big((M+2u)^{\alpha n}\big)\,\frac{x^n}{n!} \\
&= L\left(\sum_{n=0}^{\infty} \frac{\big((M+2u)^\alpha x\big)^n}{n!}\right) \\
&= L\big(e^{(M+2u)^\alpha x}\big)
\end{aligned}$$

□

Lemma 2.3.

$$L(e^{Mx}) = e^{-x^2}$$

Theorem 2.4. *(Exponential Generating Function of $H_n(u)$)*

$$F(u,x) \equiv \sum_{n=0}^{\infty} \frac{H_n(u)}{n!}\,x^n = e^{2ux-x^2}$$

Proof.

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{H_n(u)}{n!}\,x^n &= L(e^{(M+2u)x}) \\
&= e^{2ux} L(e^{Mx}) \\
&= e^{2ux} e^{-x^2} \\
&= e^{2ux-x^2}
\end{aligned}$$

□

3. Telephone Numbers

The nth telephone number gives the number of possible configurations of an n-person telephone network in which only two-person phone calls are possible. The sequence of all telephone numbers is denoted t_n. This sequence can be expressed by the following recurrence relation:

$$t_n = t_{n-1} + (n-1)t_{n-2},$$

$$t_0 = 1,\ t_1 = 1.$$

Proof. Consider a network of n telephone users. Suppose user 1 is not calling anyone. Then there are t_{n-1} configurations of phone conversations possible among the remaining users. Now suppose that user 1 is talking to someone. Then there are $n-1$ users who can talk to user 1. Since these two users cannot converse with any other users, there are t_{n-2} configurations possible for the remaining users. □

We wish to know more about the telephone numbers. In doing so, we would like to look at the EGF for t_n. We will translate the recurrence relation for t_n into an EGF, $T(x)$, in the following lemma.

Lemma 3.1.

$$T(x) = \sum_{n=0}^{\infty} \frac{t_n}{n!} x^n = e^{x+x^2/2}$$

Proof. Since $t_{n+1} = t_n + nt_{n-1}$,

$$\sum_{n=1}^{\infty} \frac{t_{n+1}}{n!} x^n = \sum_{n=1}^{\infty} \frac{t_n}{n!} x^n + \sum_{n=1}^{\infty} \frac{nt_{n-1}}{n!} x^n. \quad (\dagger)$$

Note:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{t_{n+1}}{n!}\, x^n &= \sum_{n=2}^{\infty} \frac{t_n}{(n-1)!}\, x^{n-1} \\ &= \left(\sum_{n=1}^{\infty} \frac{t_n}{(n-1)!}\, x^{n-1} - 1 \right) \\ &= T'(x) - 1.\end{aligned}$$

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{t_n}{n!}\, x^n &= \sum_{n=0}^{\infty} \frac{t_n}{n!}\, x^n - 1 \\ &= T(x) - 1.\end{aligned}$$

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{n t_{n-1}}{n!}\, x^n &= \sum_{n=1}^{\infty} \frac{t_{n-1}}{(n-1)!}\, x^n \\ &= x \sum_{n=1}^{\infty} \frac{t_{n-1}}{(n-1)!}\, x^{n-1} \\ &= x \sum_{n=0}^{\infty} \frac{t_n}{n!}\, x^n \\ &= xT(x).\end{aligned}$$

From these observations and (†), we have

$$T'(x) - 1 = T(x) - 1 + xT(x)$$

which gives us the seperable differential equation

$$T'(x) = (1+x)T(x).$$

Solving for $T(x)$ with the initial condition $T(0) = 1$, we find

$$T(x) = e^{x + x^2/2}$$

□

4. Relating Hermite Polynomials to Telephone Numbers

In Theorem 2.4, we used umbral methods to derive the exponential generating function of the Hermite polynomials, so, we can now discuss potential applications of exponential generating functions. One application of exponential generating functions would be to discover the relationship between Hermite polynomials and the telephone numbers. First, we show the relationship between them using their standard power series representations which are given and proven below.

Lemma 4.1.

$$H_n(u) = n! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{k!\,(n-2k)!}(2u)^{n-2k}$$

Proof.

$$\begin{aligned} H_n(u) &= L\big((M+2u)^n\big) \\ &= \sum_{k=0}^{n} \binom{n}{k} L(M^k)\,(2u)^{n-k} \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} L(M^{2k})\,(2u)^{n-2k} \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{(2k)!\,(n-2k)!}\,\frac{(-1)^k(2k)!}{k!}\,(2u)^{n-2k} \\ &= n! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{k!\,(n-2k)!}(2u)^{n-2k} \end{aligned}$$

□

Lemma 4.2.

$$t_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{2^k k!(n-2k)!}$$

Proof. First,

$$\begin{aligned}
\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{2^k k!(n-2k)!} &= \sum_{k=0}^{\lfloor n/2 \rfloor} \left(\frac{n!}{2^k k!(n-2k)!}\right)\left(\frac{2k!}{2k!}\right) \\
&= \sum_{k=0}^{\lfloor n/2 \rfloor} \left(\frac{n!}{2k!(n-2k)!}\right)\left(\frac{2k!}{2^k k!}\right) \\
&= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}(2k-1)!!
\end{aligned}$$

Now, we let k denote the number of phone calls an n-user network. Each phone call involves a single pair of people. We may have at most $\lfloor n/2 \rfloor$ pairs. So k ranges from 0 to $\lfloor n/2 \rfloor$. Supoose k is some particular integer in $[0, \lfloor n/2 \rfloor]$. There are $\binom{n}{k}$ ways to select $2k$ users to be calling. Labeling each person $p_1, p_2, \ldots, p_n$, we then have the first person, p_1, able to call $(2k-1)$ people. Similarly, p_2, can call $(2k-3)$ people, p_3 can call $(2k-5)$ people, until $p_{(2k-1)}$ can only call one other user. Hence, we have $(2k-1)!!$ ways of choosing these pairs. □

Using lemmas 4.1 and 4.2, we can now prove the following relation between $H_n(u)$ and and t_n.

Theorem 4.3 (Relating $H_n(u)$ to t_n)**.**

$$\left(\frac{i}{\sqrt{2}}\right)^n H_n\left(\frac{-i}{\sqrt{2}}\right) = t_n$$

Proof.

$$
\begin{aligned}
\left(\frac{i}{\sqrt{2}}\right)^n H_n\left(\frac{-i}{\sqrt{2}}\right) &= \frac{i^n}{(\sqrt{2})^n} n! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{k!(n-2k)!}\left(2\left(\frac{-i}{\sqrt{2}}\right)\right)^{n-2k} \\
&= \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{i^n n!(-1)^k 2^{n-2k}(-1)^{n-2k} i^{n-2k}}{(\sqrt{2})^{2n-2k} k!(n-2k)!} \\
&= \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{i^{2n-2k} 2^{n-k}(-1)^{n-k} n!}{2^{n-k} k!(n-2k)!} \\
&= \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^{n-k}(-1)^{n-k} n!}{2^k k!(n-2k)!} \\
&= \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{2^k k!(n-2k)!} \\
&= t_n
\end{aligned}
$$

□

While this proof is straightforward, having the insight to choose appropriate values for u and x may have proven to be a bit more challenging. Instead, one might wish to take a closer look at the exponential generating functions for both the Hermite polynomials and the telephone numbers in search of a relationship between their exponential generating functions. Intuitively, there seems to be a relationship between these two EGFs. In fact, we see that if we pick u' and x' (for $F(u', x')$) such that we have $2u'x' = x$ and $-(x')^2 = x^2/2$, the relationship becomes clear. The appropriate choices are $u' = -i/\sqrt{2}$ and $x' = ix/\sqrt{2}$, and we obtain the following theorem:

Theorem 4.4.

$$F\left(\frac{-i}{\sqrt{2}}, \frac{ix}{\sqrt{2}}\right) = T(x)$$

Proof.

$$F\left(\frac{-i}{\sqrt{2}}, \frac{ix}{\sqrt{2}}\right) = e^{2(-i/\sqrt{2})(ix/\sqrt{2})-(ix/\sqrt{2})^2} = e^{x+x^2/2} = T(x)$$

□

Having just shown the relationship between their exponential generating functions, we can write the exponential generating functions in their power series representations, with $2ux = x$ and $-x^2 = \frac{x^2}{2}$. So, we have

Proof.

$$\sum_{n=0}^{\infty} \frac{H_n\left(\frac{-i}{\sqrt{2}}\right)}{n!} \left(\frac{ix}{\sqrt{2}}\right)^n = \sum_{n=0}^{\infty} \frac{t_n}{n!} x^n.$$

Hence for an positive integer n,

$$\frac{H_n\left(\frac{-i}{\sqrt{2}}\right)}{n!} \left(\frac{ix}{\sqrt{2}}\right)^n = \frac{t_n}{n!} x^n.$$

Therefore,

$$\left(\frac{i}{\sqrt{2}}\right)^n H_n\left(\frac{-i}{\sqrt{2}}\right) = t_n.$$

□

We have arrived at our previous statement. Using this method instead we have a much simpler task in finding u and x that make this true. While employing our first method requires significantly more manipulation within a power series to arrive to the same conclusion, comparing the exponential generating functions instead allowed us to quickly obtain the same result.

Theorem 4.5.

$$\left(\frac{i}{\sqrt{2}}\right)^n H_n\left(\frac{-i}{\sqrt{2}}\right) = t_n$$

There is also a rather interesting relationship between t_n and the probabilists' Hermite polynomials, $He_n(u)$. In the proof of the following theorem, we use the closed formula for $He_n(u)$, which is easily derived given that $H_n(u) = 2^{n/2} He_n(u\sqrt{2})$.

Theorem 4.6. *The sum of absolute values of the coefficients of the nth (probabilist) Hermite polynomial is the nth telephone number.*

Proof. Since

$$He_n(u) = n! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{k!\,(n-2k)!} \cdot \frac{u^{n-2k}}{2^k},$$

we have that the sum of absolute values of the coefficients of $H_n(u)$ is

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \left| \frac{n!\,(-1)^k}{2^k\, k!\,(n-2k)!} \right| = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{2^k\, k!\,(n-2k)!} = t_n.$$

□

The following lemmas will be useful in proving theorem (4.9). Proofs are given by De Angelis [2].

Lemma 4.7. *If $f(M)$ is a power series, then*

$$L\big(e^{M^2\gamma} f(M)\big) = \frac{1}{\sqrt{1+4\gamma}}\, L\bigg(f\Big(\frac{M}{\sqrt{1+4\gamma}}\Big)\bigg)$$

Lemma 4.8.

$$L(e^{M^3\alpha + M\beta}) = \frac{e^{-(\beta+z)^2+\phi}}{\sqrt{1-24\alpha(\beta+z)}}\, L\big(e^{M^3\alpha/(1-24\alpha(\beta+z))^{3/2}}\big),$$

where $\phi = 2\beta z - 8\alpha\beta^3 - 24\alpha\beta^2 z + +2z^2 - 24\alpha\beta z^2 - 8\alpha z^3$,
and z is a solution of $12\alpha z^2 + (24\alpha\beta - 1)z + 12\alpha\beta^2 = 0$.

Theorem 4.9.
$$\sum_{n=0}^{\infty} H_{3n}(u)\,\frac{x^n}{n!} = \frac{e^{8v^3x+144v^4x^2}}{(1+48ux)^{1/4}}\sum_{n=0}^{\infty}\frac{(-1)^n\,(6n)!}{(3n)!\,(1+48ux)^{3n/2}}\,\frac{x^{2n}}{(2n)!},$$
where $v = (\sqrt{1+48ux}-1)/(24x)$.

Proof. By lemma (2.2),
$$\begin{aligned}\sum_{n=0}^{\infty} H_{3n}(u)\,\frac{x^n}{n!} &= L\Big(e^{(2u+M)^3x}\Big)\\ &= e^{8u^3x}L(e^{6uxM^2}e^{xM^3+12u^2xM}).\end{aligned}$$

By lemma (4.7), with $\gamma = 6ux$ and $f(M) = e^{xM^3+12xu^2M}$, we get
$$e^{8u^3x}L(e^{6uxM^2}e^{xM^3+12xu^2M}) = \frac{e^{8xu^3}}{\sqrt{1+24ux}}\,L\Big(e^{(x/(1+24ux)^{3/2})M^3+((12u^2x)/\sqrt{1+24ux})M}\Big).$$
Now by lemma (4.8), with $\alpha = \frac{x}{(1+24xu)^{3/2}}$, $\beta = \frac{12xu^2}{\sqrt{1+24xu}}$, $z = \frac{1-24\alpha\beta-\sqrt{1-48\alpha\beta}}{24\alpha}$, and $\phi = 2\beta z - 8\alpha\beta^3 - 24\alpha\beta^2 z + +2z^2 - 24\alpha\beta z^2 - 8\alpha z^3$, we have
$$\sum_{n=0}^{\infty} H_{3n}(u)\,\frac{x^n}{n!} = \frac{e^{8u^3x-(\beta+z)^2+\phi}}{\sqrt{1+24ux}\sqrt{1-24\alpha(\beta+z)}}L\Big(e^{(M^3\alpha)/(1-24\alpha(\beta+z))^{3/2}}\Big)$$
Since $z = \frac{1-24\alpha\beta-\sqrt{1-48\alpha\beta}}{24\alpha}$, we have $1-24\alpha(\beta+z) = \sqrt{1-48\alpha\beta}$. So,
$$\sum_{n=0}^{\infty} H_{3n}(u)\,\frac{x^n}{n!} = \frac{e^{8u^3x-(\beta+z)^2+\phi}}{\sqrt{(1+24ux)\sqrt{1-48\alpha\beta}}}L\Big(e^{M^3\alpha/(1-48\alpha\beta)^{3/4}}\Big).$$
Since $\alpha = \frac{x}{(1+24xu)^{3/2}}$ and $\beta = \frac{12xu^2}{\sqrt{1+24xu}}$, we obtain $\sqrt{(1+24xu)\sqrt{1-48\alpha\beta}} = (1+48xu)^{1/4}$ through simplification. Hence,
$$\sum_{n=0}^{\infty} H_{3n}(u)\,\frac{x^n}{n!} = \frac{e^{8u^3x-(\beta+z)^2+\phi}}{(1+48xu)^{\frac{1}{4}}}L\Big(e^{(M^3\alpha)/(1-48\alpha\beta)^{3/4}}\Big).$$
By the definition of e^x, we have
$$\begin{aligned}L\Big(e^{(M^3\alpha)/(1-48\alpha\beta)^{3/4}}\Big) &= L\Bigg(\sum_{n=0}^{\infty}\frac{1}{n!}\cdot\Big(\frac{M^3\alpha}{(1-48\alpha\beta)^{3/4}}\Big)^n\Bigg)\\ &= L\Bigg(\sum_{n=0}^{\infty}M^{3n}\frac{\alpha^n}{n!\,(1-48\alpha\beta)^{3n/4}}\Bigg),\end{aligned}$$

and because L is linear over umbrae, we get

$$L\Big(\sum_{n=0}^{\infty} M^{3n}\frac{\alpha^n}{n!\,(1-48\alpha\beta)^{3n/4}}\Big)=\sum_{n=0}^{\infty} L\big(M^{3n}\big)\frac{\alpha^n}{n!\,(1-48\alpha\beta)^{3n/4}}.$$

Since $L\big(M^{2n+1}\big)=0$ and $L\big(M^{2n}\big)=\frac{(-1)^n(2n)!}{n!}$,

$$\begin{aligned}\sum_{n=0}^{\infty} L\big(M^{3n}\big)\frac{\alpha^n}{n!\,(1-48\alpha\beta)^{3n/4}} &= \sum_{n=0}^{\infty} L\big(M^{6n}\big)\frac{\alpha^{2n}}{(2n)!\,(1-48\alpha\beta)^{3n/2}}\\ &= \sum_{n=0}^{\infty}\frac{(-1)^{3n}(6n)!}{(3n)!}\cdot\frac{\alpha^{2n}}{(2n)!(1-48\alpha\beta)^{3n/2}}.\end{aligned}$$

Now, since $\alpha=\frac{x}{(1+24xu)^{3/2}}$ and $\beta=\frac{12xu^2}{\sqrt{1+24xu}}$, we have

$$\sum_{n=0}^{\infty}\frac{(-1)^{3n}(6n)!}{(3n)!}\cdot\frac{\alpha^{2n}}{(2n)!(1-48\alpha\beta)^{3n/2}}=\sum_{n=0}^{\infty}\frac{(-1)^n\,(6n)!}{(3n)!\,(1+48ux)^{3n/2}}\,\frac{x^{2n}}{(2n)!}.$$

Thus,

$$\sum_{n=0}^{\infty} H_{3n}(u)\,\frac{x^n}{n!}=\frac{e^{8u^3x-(\beta+z)^2+\phi}}{(1+48xu)^{\frac{1}{4}}}\sum_{n=0}^{\infty}\frac{(-1)^n\,(6n)!}{(3n)!\,(1+48ux)^{3n/2}}\,\frac{x^{2n}}{(2n)!}$$

Since $\beta=\frac{12xu^2}{\sqrt{1+24xu}}$, $z=\frac{1-24\alpha\beta-\sqrt{1-48\alpha\beta}}{24\alpha}$, and
$\phi=2\beta z-8\alpha\beta^3-24\alpha\beta^2z++2z^2-24\alpha\beta z^2-8\alpha z^3$, through algebraic manipulation in Mathematica we have

$$\begin{aligned}8xu^3-(\beta+z)^2+\phi &= \frac{1+72xu+864x^2u^2-\big(\frac{(1152x^2+72xu+1)\sqrt{1+48xu}}{1+24xu}\big)}{864x^2}\\ &= \frac{1+72xu+864x^2u^2-\big(\frac{(1+24xu)(1+48xu)\sqrt{1+48xu}}{1+24xu}\big)}{864x^2}\\ &= \frac{1+72xu+864x^2u^2-(1+48xu)^{\frac{3}{2}}}{864x^2}\\ &= 8v^3x+144v^4x^2,\end{aligned}$$

where $v=(\sqrt{1+48ux}-1)/(24x)$.

Therefore,

$$\sum_{n=0}^{\infty} H_{3n}(u)\,\frac{x^n}{n!} = \frac{e^{8v^3x+144v^4x^2}}{(1+48ux)^{1/4}} \sum_{n=0}^{\infty} \frac{(-1)^n\,(6n)!}{(3n)!\,(1+48ux)^{3n/2}}\,\frac{x^{2n}}{(2n)!},$$

where $v = (\sqrt{1+48ux}-1)/(24x)$. □

Given theorems 4.5 and 4.9, we are brought to the following corollary.

Corollary 4.10.

$$\sum_{n=0}^{\infty} t_{3n}\,\frac{x^n}{n!} = \frac{e^{-4w^3x\sqrt{2}-72w^4x^2}}{(1+24x)^{1/4}} \sum_{n=0}^{\infty} \frac{(6n)!}{(3n)!\,(1+24x)^{3n/2}}\,\frac{x^{2n}}{(2n)!\,2^n},$$

where

$$w = \frac{\sqrt{2}(\sqrt{1+24x}-1)}{24x}.$$

Proof. From theorem 4.5, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} t_{3n}\,\frac{x^n}{n!} &= \sum_{n=0}^{\infty} H_{3n}\left(\frac{-i}{\sqrt{2}}\right)\frac{\left(\frac{ix}{\sqrt{2}}\right)^n}{n!} \\
&= \frac{e^{8v^3(ix/\sqrt{2})+144v^4(ix/\sqrt{2})^2}}{(1+48(\frac{-i}{\sqrt{2}})(\frac{ix}{\sqrt{2}}))^{1/4}} \sum_{n=0}^{\infty} \frac{(-1)^n\,(6n)!}{(3n)!\,(1+48(\frac{-i}{\sqrt{2}})(\frac{ix}{\sqrt{2}}))^{3n/2}}\,\frac{(\frac{ix}{\sqrt{2}})^{2n}}{(2n)!} \\
&= \frac{e^{4v^3ix\sqrt{2}-72v^4x^2}}{(1+24x)^{1/4}} \sum_{n=0}^{\infty} \frac{(6n)!}{(3n)!\,(1+24x)^{3n/2}}\,\frac{x^{2n}}{(2n)!\,2^n},
\end{aligned}$$

where

$$\begin{aligned}
v &= \frac{\sqrt{1+48\left(\frac{-i}{\sqrt{2}}\right)\left(\frac{ix}{\sqrt{2}}\right)}-1}{24\left(\frac{ix}{\sqrt{2}}\right)} \\
&= \frac{\sqrt{2}(\sqrt{1+24x}-1)}{24ix}.
\end{aligned}$$

Letting $w = iv$, we have

$$\sum_{n=0}^{\infty} t_{3n}\,\frac{x^n}{n!} = \frac{e^{-4w^3x\sqrt{2}-72w^4x^2}}{(1+24x)^{1/4}} \sum_{n=0}^{\infty} \frac{(6n)!}{(3n)!\,(1+24x)^{3n/2}}\,\frac{x^{2n}}{(2n)!\,2^n},$$

where

$$w = \frac{\sqrt{2}(\sqrt{1+24x}-1)}{24x}.$$

□

References

[1] I. M. Gessel, *Applications of the Classical Umbral Calculus.Algebra univers.*, **49** (2003), 397–434.
[2] V. De Angelis, Class Lecture, *The Umbral Calculus,* Louisiana State University, Baton Rouge, LA, Summer 2013.