STABILITY ANALYSIS OF AN SIR EPIDEMIC MODEL

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ABSTRACT. To properly understand the behavior of the SIR model we must first perform a complete stability analysis of the model. This task includes finding the equilibrium points, the upper and lower bounds of the System, and applying appropriate theorems to determine local or global stability. We must also determine whether the equilibrium point is asymptotically stable or simply stable. For our model we found that the system is globally asymptotically stable.

1. Background Information

Consider the model

$$S'(t) = \Lambda - \beta S \frac{1}{N} - \mu S$$

$$I'(t) = \beta S \frac{1}{N} - (\mu + \gamma) I \tag{1.1}$$

$$R'(t) = \gamma I - \mu R,$$

where

$$N(t) = S(t) + I(t) + R(t). \tag{1.2}$$

S, I, and R represent the densities of individuals subject to a disease who are susceptible, infective, and removed, respectively, and N is the population. Λ is a fixed number of individuals who join or arrive into the susceptible class per unit, μ is the per capita death rate, and γ is the per capita recovery rate. Given this information, we will perform a complete stability analysis of 1.1.

2. Converting from Nonautonomous to Autonomous Systems of Equations

To perform the stability analysis of System 1.1, we will employ an autonomous system of equations instead of a nonautonomous system.

Definition 2.1. A function $f(x)$ satisfies a **Lipschitz** condition on the interval I if there exitst a constant L such that $|f(r)-f(s)| \leq L|r-s|$ for all $r, s, \in I$. Any continuous function is uniformly bounded over a finite integral $f(r)-f(s)=f'(\epsilon)(r-s)$.

Definition 2.2. Consider the following systems:

$$\frac{d(X)}{dt}=F(t,X) \tag{2.1}$$

$$\frac{d(Y)}{dt}=G(X), \tag{2.2}$$

where F and G are continuous and locally Lipschitz in X and Y, respectively, for all X and Y in $\mathbb{R}^n$, and solutions exist globally. System 2.1 is called **asymptotically autonomous** with limit System 2.2 if $F(t,X) \rightarrow G(X)$ as $t \rightarrow \infty$ for X in $\mathbb{R}^n$.

Theorem 2.3. *If solutions of System 2.1 are bounded and the equilibrium of X of System 2.2 is globally asymptotically stable, then any solution $X(t)$ of System 2.1 satisfies $X(t) \rightarrow X$ as $t \rightarrow \infty$.*

System 1.1 is clearly nonautonomous. Therefore, we must convert it to a system of autonomous equations. To do this we must determine if the system is bounded, so we must find the lower and upper bounds of the equations.

2.1. Lower Bound.

$N(t)$ is the population at any given time. The population of the system must always be positive because a negative value for a population does not make sense. Therefore, $N>0$. This implies that $S(t)>0$ and $I(t)>0$ and $R(t) \geq 0$. Thus, the lower bound for S, I, and R is 0.

2.2. Upper Bound.

We use Equation 1.2 yet again, this time taking the derivative of each side. When we add the equations of System 1.1, we obtain the solution $N'(t)=\Lambda-\mu N(t)$. Then we simplify and solve the differential equation for N. In doing so we obtain the solution $N=\frac{\Lambda}{\mu}+\frac{c}{e^{\mu t}}$. We are searching for the limit system, so we must take the limit of the equation. Therefore, $lim_{t\rightarrow\infty}(\frac{\Lambda}{\mu}+\frac{c}{e^{\mu t}})=\frac{\Lambda}{\mu}$ when $t \rightarrow \infty$. Consequently, $N \approx \frac{\Lambda}{\mu}$ for large values of t. Thus, System 1.1 has a limit system. We know that S, I, and R are separately less than or equal to N because their sum is equal to N. This implies that $S \leq \frac{\Lambda}{\mu}+\frac{c}{e^{\mu t}}$ and $I \leq \frac{\Lambda}{\mu}+\frac{c}{e^{\mu t}}$ and $R \leq \frac{\Lambda}{\mu}+\frac{c}{e^{\mu t}}$. Therefore, the upper bound for S, I, and R is $\frac{\Lambda}{\mu}+\frac{c}{e^{\mu t}}$.

2.3. **Confirming Conversion to an Autonomous System.** Now that we have found the upper and lower bounds of the equations and have solved the differential equations for N, we can apply Theorem 2.2. Thus, we may consider the autonomous limit system instead of the original nonautonomous system.

3. Equilibrium Point

We now need to find the equilibrium points for these equations.

Definition 3.1. Given an equation $\frac{dx}{dt} = f(x)$, the point x^* is an **equilibrium point** if $f(x^*) = 0$.

We can get the equilibrium points by setting the equations in System 1.1 equal to zero and solving the system for S, I, and R. By doing so, we obtain

$$S = \frac{\Lambda^2}{\mu(\beta + \Lambda)}$$

$$I = \frac{\beta\Lambda}{(\beta + \Lambda) + (\mu + \gamma)}$$

$$R = \frac{\gamma\beta\Lambda}{\mu(\beta + \Lambda)(\mu + \gamma)}.$$

We know that this is the only equilibrium point because the equation for S is linear. Therefore, the equilibrium point is $(\frac{\Lambda^2}{\mu(\beta+\Lambda)}, \frac{\beta\Lambda}{(\beta+\Lambda)+(\mu+\gamma)}, \frac{\gamma\beta\Lambda}{\mu(\beta+\Lambda)(\mu+\gamma)})$.

4. Reducing to a system of two equations

We next reduce the difficulty of performing the stability analysis by reducing System 1.1 to fewer equations. This can be done, once again, by employing Equation 1.2. We then substitute this into System 1.1 to obtain the new system of equations:

$$S' = \Lambda - \frac{\beta S}{N} - \mu S,$$

$$I' = \frac{\beta S}{N} - (\mu + \gamma)I,$$

$$R' = I(\gamma + \mu) - \mu(N - S).$$

It is clear that only two variables are listed in this system of three equations. Therefore, the last equation, R', may be disregarded.

5. Stability Analysis

Now that we have simplified our system down to two Autonomous equations, we can analyze the stability using the proper Lyapunov function.

Definition 5.1. A function $V(x, y)$ is said to be **positive definite** on a region D containing the origin if for all $(x, y) \neq 0$, $V(x, y) > 0$. $V(x, y)$ is said to be **negative definite** on a region D containing the origin if for all $(x, y) \neq 0$, $V(x, y) < 0$.

Definition 5.2. A function $V(x, y)$ is said to be a **Lyapunov Function** on an open region D if the function is continuous, positive definite, and has continuous first-order partial derivatives on D.

The derivative of V with respect to the system $\frac{dx}{dt} = f(x, y)$ and $\frac{dy}{dt} = g(x, y)$ is defined as

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt}.$$

Theorem 5.3. *If there exists a Lyapunov function $V(x, y)$, dependent on a system $\frac{dx}{dt} = f(x, y)$ and $\frac{dy}{dt} = g(x, y)$ with equilibrim point $(x, y) = (0, 0)$, and $\frac{dV}{dt}$ is negative definite on an open region D containing the origin, then the zero solution of the system is asymptotically stable.*

When D encompasses all possible values of (x, y) and follows all of the specified criteria above, the projected stability of the system is said to be global.

A possible Lyaponov function that is very common is $V = x^2 + y^2$. We are interested in showing not only local, but global asymptotic stability. This means that our chosen V will have to satisfy the criteria of a Lyaponov function over the entire region $D = (0, \infty)$x$(0, \infty)$ for which SxI is defined. This criteria includes

i) $V(0, 0) = (0, 0)$
ii) $V(x, y) > 0 \; \forall \; (x, y) \neq (0, 0)$ on D (positive definite).

However, this Lyapunov function only works for those systems with an equilibrium point set at the origin, and as can be seen looking back at our specified region D, the origin is not included. This forces us to use a change of variables in order to utilize the function $V = x^2 + y^2$.

We take

$$x = S - \frac{\Lambda^2}{\mu(\beta+\Lambda)},$$

$$y = I - \frac{\beta\Lambda}{(\beta+\Lambda)(\mu+\gamma)}.$$

So

$$V = (S - \frac{\Lambda^2}{\mu(\beta+\Lambda)})^2 + (I - \frac{\beta\Lambda}{(\beta+\Lambda)(\mu+\gamma)})^2.$$

When redefining our D function for (x,y), it is shifted to the left, leaving us with $D_s = (-\frac{\Lambda^2}{\mu(\beta+\Lambda)}, \infty) \times (-\frac{\beta\Lambda}{(\beta+\Lambda)(\mu+\gamma)}, \infty)$, an open interval including the origin.

We now check the derivative of our V function:

$$\frac{dV}{dt} = \frac{\partial V}{\partial S}\frac{dS}{dt} + \frac{\partial V}{\partial I}\frac{dI}{dt}$$

$$= 2(S - \frac{\Lambda^2}{\mu(\beta+\Lambda)})(\Lambda - \beta S\frac{1}{N} - \mu S) + 2(I - \frac{\beta\Lambda}{(\beta+\Lambda)(\mu+\gamma)})(\beta S\frac{1}{N} - (\mu+\gamma)I)$$

$$= -2[(\frac{\mu(\beta+\Lambda)}{\Lambda^2})(S - \frac{\Lambda^2}{\mu(\beta+\Lambda)})^2 + (\frac{(\beta+\Lambda)(\mu+\gamma)}{\beta\Lambda})(I - \frac{\beta\Lambda}{(\beta+\Lambda)(\mu+\gamma)})^2].$$

The terms inside the brackets will only ever yield non-negative numbers since all of our terms are positive and the only portions containing subtraction have been squared. So the overall sign of $\frac{dV}{dt}$ is determined by the factor of -2 outside the brackets. Furthermore, the only point that will make this equation equal to zero is the equilibrium point x=0, y=0, which corresponds to the values $S = \frac{\Lambda^2}{\mu(\beta+\Lambda)}$, $I = \frac{\beta\Lambda}{(\beta+\Lambda)(\mu+\gamma)}$. So $\frac{dV}{dt}$ is negative definite in $\mathbb{R}^2$, which implies that $\frac{dV}{dt}$ is negative definite on our specified interval D. Therefore, our system is globally asymptotically stable.

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