PUTZER'S METHOD VIA THE $\mathcal{Z}$-TRANSFORM

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Abstract. For a system of first order linear difference equations, $y(k+1) = Ay(k)$, Putzer's algorithm computes the nxn coefficient matrix A raised to some k^{th} power, where k is a nonnegative integer, by using the eigenvalues of A. The procedure is recursive. Using the $\mathcal{Z}$-transform, which removes recursions in difference equations, we redefine Putzer's algorithm in a closed form via the $\mathcal{Z}$-transform.

1. Introduction

Given an $n \times n$ matrix A, let

$$\mathbf{y}(k) = \begin{bmatrix} y_1(k) \\ y_2(k) \\ \vdots \\ y_n(k) \end{bmatrix}$$

be the column vector containing entries $y_1(k), ..., y_n(k)$. The solution to the system of equations with an initial condition

$$\mathbf{y}(k+1) = A\mathbf{y}(k), \quad \mathbf{y}(0) = \mathbf{y_0}$$

is

$$\mathbf{y}(k) = A^k \mathbf{y_0}.$$

The struggle here is to compute A^k. There are known methods including transforming A into its Jordan Canonical Form, Fulmer's Method, and the use of $\mathcal{Z}$-transforms on the matrix A itself. We use Putzer's Method which is a recursive algorithm using the eigenvalues of A. The recursive nature of the algorithm becomes difficult when calculating A^k, where k is a large positive integer. Via the $\mathcal{Z}$-transform, we rewrite Putzer's Method in a closed form. The first section reviews the Cayley-Hamilton theorem and the Uniqueness and Existence theorem. Next we cover Putzer's Method and the $\mathcal{Z}$-transform. We conclude with Putzer's method redfined using the $\mathcal{Z}$-transform.

2. Cayley-Hamilton and the Existence and Uniqueness Theorems

We state the following theorem without proof.

Theorem 1. *Given an $n \times n$ matrix, A, define the characteristic polynomial as*

$$p_A(\lambda) = det(A - \lambda I).$$

Then the matrix A satisfies its own characteristic polynomial, that is,

$$p_A(A) = 0.$$

Example 2. *Define*

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}.$$

Its characteristic polynomial is given by

$$\begin{aligned} p_A(\lambda) &= \det(A - \lambda I) \\ &= \begin{vmatrix} 2-\lambda & 3 \\ 4 & 5-\lambda \end{vmatrix} \\ &= (2-\lambda)(5-\lambda) - (3)(4) \\ &= \lambda^2 - 7\lambda - 2. \end{aligned}$$

The Cayley-Hamilton theorem claims

$$p_A(A) = A^2 - 7A - 2I = 0.$$

Now

$$\begin{aligned} A^2 - 7A - 2I &= \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} - 5\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} - \begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

A satisfies its characteristic polynomial.

The following theorem guarantees the existence and uniqueness of solutions to the above system of difference equations.

Theorem 3. *For each $k_0 \in \{a, a+1, \ldots\}$ and each n-vector y_0 the equation*

$$y(k+1) = A(t)y(k) + f(k)$$

has a unique solution $y(k)$ defined for $k \in \{k_0, k_0+1, \ldots\}$, so that $y(k_0) = y_0$.

Putzer's Method, which will be discussed in the next section solves a difference equation of the form $y(k+1) = Ay(k)$, the solution of which is given in the following theorem.

Theorem 4. *The solution to the difference equation $y(k+1) = Ay(k)$ is given by $y(k) = A^k y_0$, where $y(0) = y_0$ is the initial condition.*

Proof. That a solution exists and is unique follows from Theorem 3. Observe that

$$\begin{aligned} y(1) &= Ay(0) \\ y(2) &= Ay(1) &= A^2 y(0) \\ y(3) &= Ay(2) &= A^3 y(0) \\ &\vdots \end{aligned}$$

and inductively we see that $y(k) = A^k y(0)$.

□

3. Putzer's Method

Theorem 5. *Given the system of first order linear difference equations $\mathbf{y}(k+1) = A\mathbf{y}(k)$ with the initial condition vector $\mathbf{y}(0) = \mathbf{y_0}$ and eigenvalues listed as many times as their multiplicity $\lambda_1, \ldots, \lambda_n$, then the solution $\mathbf{y}(k) = A^k \mathbf{y_0}$ can be written in the form*

$$\mathbf{y}(k) = \sum_{i=0}^{n-1} c_{i+1}(k) M_i \mathbf{y_0} = A^k \mathbf{y_0}$$

where

$$\begin{aligned} M_0 &= I \\ M_i &= (A - \lambda_i)(A - \lambda_{i-1}) \ldots (A - \lambda_1), \quad 1 \leq i \leq n \end{aligned}$$

and

$$\begin{aligned} c_1(k+1) &= \lambda_1 c_1(k), \quad c_1(0) = 1 \\ c_i(k+1) &= \lambda_i c_i(k) + c_{i-1}(k), \quad c_i(0) = 0, \quad 2 \leq i \leq n. \end{aligned}$$

Proof. By the Caley-Hamilton theorem, A^k can be written as a linear combination of $I, A, A^2, \ldots, A^{n-1}$, provided that A is an n x n matrix, as we will now demonstrate. Given the characteristic polynomial

$$\begin{aligned} p_A(\lambda) &= a_n\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_2\lambda^2 + a_1\lambda + a_0, \\ p_A(A) &= a_nA^n + a_{n-1}A^{n-1} + \ldots + a_2A^2 + a_1A + a_0I = 0 \end{aligned}$$

by the Cayley-Hamilton theorem. Now

$$\begin{aligned} A^n &= -\frac{a_{n-1}}{a_n}A^{n-1} - \ldots - \frac{a_2}{a_n}A^2 - \frac{a_1}{a_n}A - \frac{a_0}{a_n}I \\ &= b_nA^{n-1} + \ldots + b_3A^2 + b_2A + b_1I. \end{aligned}$$

This implies that every power of A can also be written as a linear combination of $I, A, A^2, \ldots, A^{n-1}$, since every power of A can be found by multiplying the above equation successively by A and substituting the representation of A^n into it to reduce the powers, as seen below. Observe that

$$\begin{aligned} A^{n+1} &= AA^n \\ &= b_nA^n + \ldots + b_3A^3 + b_2A^2 + b_1A \\ &= b_n(b_nA^{n-1} + \ldots + b_3A^2 + b_2A + b_1a_nI) + \ldots + b_3A^3 + b_2A^2 + b_1A \\ &= c_nA^{n-1} + \ldots + c_3A^2 + c_2A + c_1I \end{aligned}$$

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A, with each eigenvalue repeated as many times as its multiplicity. Let

$$\begin{aligned} M_0 &= I \\ M_i &= (A - \lambda_iI)M_{i-1}, \quad 1 \leq i \leq n. \end{aligned}$$

From the Cayley-Hamilton Theorem, we deduce that $M_n = 0$, since

$$M_n = p_A(A) = 0.$$

The equation for the M_i implies that each A^k can be written as a linear combination of $M_0, \ldots, M_n$ since it simply contains powers of A. So now we have

$$A^k = \sum_{i=0}^{n-1} c_{i+1}(k)M_i$$

for $k \geq 0$, where the $c_{i+1}(k)$ are to be determined. Since $A^{k+1} = AA^k$,

$$\begin{aligned}\sum_{i=0}^{n-1} c_{i+1}(k+1)M_i &= A\sum_{i=0}^{n-1} c_{i+1}(k)M_i \\ &= \sum_{i=0}^{n-1} c_{i+1}(k)[M_{i+1} + \lambda_{i+1}M_i] \\ &= \sum_{i=1}^{n-1} c_i(k)M_i + \sum_{i=0}^{n-1} c_{i+1}(k)\lambda_{i+1}M_i\end{aligned}$$

where we have replaced i by $i-1$ in the first sum and used the fact that $M_n = 0$. The preceding equation is satisfied if the $c_i(k), (i = 1, \ldots, n)$ are chosen to satisfy the system:

$$\begin{bmatrix} c_1(k+1) \\ \vdots \\ c_n(k+1) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 1 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 1 & \lambda_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \lambda_n \end{bmatrix} \begin{bmatrix} c_1(k) \\ \vdots \\ c_n(k) \end{bmatrix}.$$

Since $A^0 = I = c_1(0)I + \ldots + c_n(0)M_{n-1}$, the system is subject to the initial condition vector

$$y(0) = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

From the matrix equation above, we can write the coefficients in the form

$$\begin{aligned} c_1(k+1) &= \lambda_1 c_1(k) \\ c_2(k+1) &= \lambda_2 c_2(k) + c_1(k) \\ &\vdots \\ c_n(k+1) &= \lambda_n c_n(k) + c_{n-1}(k). \end{aligned}$$

Here we can see the amount of recursion needed to achieve the coefficients for the matrices M_i. The recursion is tedious to work with. Later in the paper we will discuss how to remove the recursion and simplify the method using the $\mathcal{Z}$-transform.

□

Example 6. *Consider the matrix* $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$. *Use Putzer's Algorithm to give a formula for* A^k.

We begin by finding the characteristic polynomial, by finding $det(A - \lambda I)$. *This yields the equation*

$$p_A(\lambda) = \lambda^2 + 3\lambda + 2$$

which produces the eigenvalues $\lambda_1 = -1$, $\lambda_2 = -2$. *From the definition of* M_i

$$M_0 = I$$

using λ_1 *for the eigenvalue in the equation* $M_i = (A - \lambda_i I)$

$$M_1 = (A + I) = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}.$$

Using the system described earlier we can get the coefficients for the matrices M_0 *and* M_1. *Solving for the first coefficient yields*

$$c_1(k+1) = -1c_1(k)$$

which has the solution $c_1(k) = (-1)^k$. *Solving for the second coefficient yields*

$$c_2(k+1) = -2c_2(k) + c_1(k)$$

which has the solution $c_2(k) = -(-2)^k + (-1)^k$. *Plugging everything into the formula for Putzer's Method*

$$\begin{aligned} A^k &= c_1 M_0 + c_2 M_1 \\ &= (-1)^k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + [(-1)^k - (-2)^k] \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 2(-1)^k - (-2)^k & (-1)^k - (-2)^k \\ -2(-1)^k + 2(-2)^k & -(-1)^k + 2(-2)^k \end{bmatrix}. \end{aligned}$$

4. $\mathcal{Z}$-Transform

The $\mathcal{Z}$-transform of a function $y(k)$ is defined as

$$\mathcal{Z}\{y(k)\}(z) = \sum_{n=0}^{\infty} \frac{y(k)}{z^k}$$

where $k \in \mathbb{Z}^+$.
We will utilize the following $\mathcal{Z}$-transforms

$$\mathcal{Z}(1) = \frac{z}{z-1}$$

$$\mathcal{Z}(a^k) = \frac{z}{z-a}.$$

An example of how we obtain these equations for the $\mathcal{Z}$-transform is shown here.

Example 7.

$$\begin{aligned}\mathcal{Z}(1)\{z\} &= \sum_{n=0}^{\infty} \frac{1}{z^k} \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^k\end{aligned}$$

Now that we have rewritten our transform into something more familiar, we can use the fact that

$$\sum_{k=0}^{\infty} ar^k = \frac{1}{1-r}$$

when $r < 1$ *and converges*

So our transform is equal to,

$$\frac{1}{1-\frac{1}{z}} = \frac{z}{z-1}.$$

We use the same approach in the previous example for the $\mathcal{Z}$-transform of a^k,

$$\begin{aligned}\mathcal{Z}(a^k)\{z\} &= \sum_{n=0}^{\infty} \frac{a^k}{z^k} \\ &= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^k\end{aligned}$$

giving

$$\mathcal{Z}(a^k) = \frac{z}{z-a}.$$

5. Applied $\mathcal{Z}$-Transform

The difference equation $y(k+1) = Ay(k)$ has solution $y(k) = A^k y(0)$. When we take the $\mathcal{Z}$-transform of the solution we get $Y(z) = \mathcal{Z}(A^k)y(0)$.

If we take the $\mathcal{Z}$-transform of the difference equation we get

$$\begin{aligned} \mathcal{Z}(y(k+1)) &= \mathcal{Z}(Ay(k)) \\ zY(z) - zy(0) &= AY(z) \\ zY(z) - AY(z) &= zy(0) \\ (z - IA)Y(z) &= zy(0) \\ Y(z) &= zy(0)(zI - A)^{-1} \end{aligned}$$

By equating our two forms of $Y(z)$ we get

$$\mathcal{Z}(A^k) = z(zI - A)^{-1}.$$

Let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of A. Define a sequence of rational functions $C_i(z)$ and matrices M_i for $0 \leq i \leq n$ by

$$C_0(z) = z$$

$$C_i(z) = z(z - \lambda_i)^{-1} \cdots (z - \lambda_n)^{-1}$$

and

$$M_0 = I$$

$$M_i = (A - \lambda_i I) \cdots (A - \lambda_1 I)$$

Now that we have defined the above, we can find a formula for $z(zI - A)^{-1}$, the resolvent matrix.

Theorem 8. *With $C_i(z)$ and M_i defined as above*

$$\mathcal{Z}(A^k) = z(zI - A)^{-1} = \sum_{i=0}^{n-1} C_{i+1}(z) M_i.$$

Proof. We can denote $R(z)$ to be the right hand side of the equation stated above. It suffices to show that $z^{-1}(zI - A)R(z) = I$. Observe

$$\begin{aligned} z^{-1}(zI - A)C_{i+1}(z)M_i &= z^{-1}(((zI - \lambda_{i+1}I) - (A - \lambda_{i+1}I))C_{i+1}(z)M_i) \\ &= z^{-1}((z - \lambda_{i+1})C_{i+1}(z)M_i - C_{i+1}(z)(A - \lambda_{i+1}I)M_i) \\ &= z^{-1}(C_i(z)M_i - C_{i+1}(z)M_{i+1}) \end{aligned}$$

where in the last line we have used that $(A - \lambda_{i+1} I)M_i = M_{i+1}$ and that $(z - \lambda_{i+1})C_i(z) = C_{i+1}(z)$. Now,

$$\begin{aligned}
z^{-1}(zI - A)R(z) &= \sum_{i=0}^{n-1} z^{-1}(zI - A)C_{i+1}(z)M_i \\
&= \sum_{i=0}^{n-1} z^{-1}C_i(z)M_i - C_{i+1}(z)M_{i+1} \\
&= z^{-1}(C_0(z)M_0 - C_n(z)M_n) \\
&= z^{-1}zI \\
&= I
\end{aligned}$$

where the sum telescoped and $C_0(z) = z$ and $M_n = 0$ by the Cayley-Hamilton theorem.

□

We now formulate the coefficients from Putzer's Method via the $\mathcal{Z}$-transform, removing the need to solve them recursively.

Theorem 9.

$$A^k = \sum_{i=0}^{n-1} \mathcal{Z}^{-1}(C_{i+1}(z))M_i = \sum_{i=0}^{n-1} c_{i+1}(k)M_i.$$

Proof. Observe that we have taken the inverse $\mathcal{Z}$-transform of the previous theorem. It suffices to show that

$$\mathcal{Z}^{-1}(C_{i+1}(z)) = c_{i+1}(k).$$

Recall the original formulation of Putzer's Method defined the coefficients according to

$$\begin{aligned}
c_1(k+1) &= \lambda_1 c_1(k), \quad c_1(0) = 1 \\
c_i(k+1) &= \lambda_i c_i(k) + c_{i-1}(k), \quad c_i(0) = 0, \quad 2 \leq i \leq n.
\end{aligned}$$

Taking the $\mathcal{Z}$-transform,

$$\begin{aligned}
zC_1(z) - zc_1(0) &= \lambda_1 C_1(z) \\
(z - \lambda_1)C_1(z) - z &= 0 \\
C_1(z) &= \frac{z}{(z - \lambda_1)}
\end{aligned}$$

and

$$\begin{aligned} zC_i(z) - zc_i(0) &= C_{i-1} + \lambda_i C_i(z) \\ (z-\lambda_i)C_i(z) &= C_{i-1}(z) \\ C_i(z) &= (z-\lambda_i)^{-1}C_{i-1}(z). \end{aligned}$$

This is exactly how we defined the coefficients $C_i(z)$, therefore,

$$\mathcal{Z}^{-1}(C_{i+1}(z)) = c_{i+1}(k).$$

□

We have eliminated the recursion from Putzer's Method and replaced it with taking the inverse $\mathcal{Z}$-transforms of the coefficients. We conclude with an example.

Example 10. *Consider the system of equations $y(k+1) = Ay(k)$ with*

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, y(0) = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

We begin by finding the characteristic polynomial

$$p_A(\lambda) = |A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{vmatrix} = (\lambda+1)(\lambda+2) = 0$$

giving

$$\lambda_1 = -1, \lambda_2 = -2.$$

Once we have the eigenvalues, we can find M_0 and M_1

$$M_0 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, M_1 = [A - \lambda_1 I] = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}.$$

Now we find the coefficients by using the $\mathcal{Z}$-transform

$$\begin{aligned} c_1(k) &= \mathcal{Z}^{-1}\{\frac{z}{z+1}\} = (-1)^k \\ c_2(k) &= \mathcal{Z}^{-1}\{\frac{z}{(z+1)(z+2)}\} \\ &= \mathcal{Z}^{-1}\{z\left(\frac{1}{z+1} - \frac{1}{z+2}\right)\} \\ &= (-1)^k - (-2)^k. \end{aligned}$$

Substituting into the equation from Putzer's Method

$$\begin{aligned} A^k &= c_1(k)M_0 + c_2(k)M_1 \\ &= \left((-1)^k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + ((-1)^k - (-2)^k) \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}\right). \end{aligned}$$

Which leads us to our final solution

$$\begin{bmatrix} 2(-1)^k - (-2)^k & (-1)^k - (-2)^k \\ -2(-1)^k + 2(-2)^k & -(-1)^k + 2(-2)^k \end{bmatrix}.$$

References

[1] Adkins, W.A., Davidson, M.G., *Putzer's Algorithm for e^{At} via the Laplace Transform*, Volume 83, Number 4, October 2010

[2] Kelley, Walter G. and Peterson, Allan C.*Difference Equations: An Introduction with Applications,* Harcourt/Academic Press, San Diego, CA, second addition, 2001.