

Legal Configurations of the 15-Puzzle

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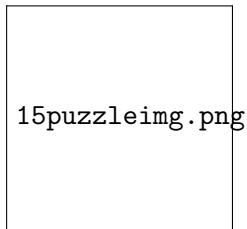
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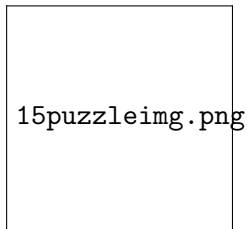
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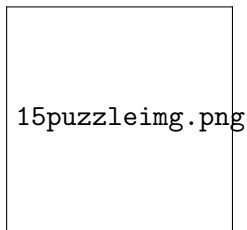
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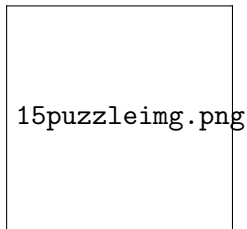
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- Our objective



Permutations

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Example

Consider the set $A = \{1, 2, 3, 4, 5, 6\}$. Then the permutation P ,

$$P = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 5 & 2 & 3 & 6 \end{pmatrix}$$

changes 1 to 4, 2 to 1, 3 to 5, 4 to 2, 5 to 3, and *fixes*, or leaves unchanged, the element 6.

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Example

Therefore, we can also write P as

$$P = (3\ 5)(1\ 4\ 2)$$

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- The set of all even permutations of n elements is denoted by A_n

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Theorem

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Lemma

The identity I , the permutation which fixes all elements, is even.

Proof of Parity Theorem

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The left hand side is a product of $s + t$ transpositions. Since Lemma 11 says that the identity on the right hand side is even, s and t must have the same parity.



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- The alternating group A_n under composition

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Theorem

$$|O_n| = |A_n| = \frac{n!}{2}.$$

Introduce Puzzle

Permutations performed on the puzzle are performed on positions and not on contents.

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4	3	2	1
5	6	7	8
12	11	10	9
13	14	15	16

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→	→	→	
↑	←	←	←
→	→	→	↑
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- The moves along this path are legal.

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There are *special* moves not along the path that are still legal. We will denote these moves $S_{i,j}$

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 S_{11,14} &= (16\ 15)(15\ 14)(14\ 13)(13\ 12)(12\ 11)(11\ 14)(14\ 15)(15\ 16) \\
 &= (16)(15)(14)(13\ 12\ 11) \\
 &= (13\ 12\ 11)
 \end{aligned}$$

Nine Moves

The following are all the legal moves not along the path:

$$\begin{aligned}S_{9,16} &= (15\ 14\ 13\ 12\ 11\ 10\ 9) \\S_{10,15} &= (14\ 13\ 12\ 11\ 10) \\S_{11,14} &= (13\ 12\ 11) \\S_{7,10} &= (9\ 8\ 7) \\S_{6,11} &= (10\ 9\ 8\ 7\ 6) \\S_{5,12} &= (11\ 10\ 9\ 8\ 7\ 6\ 5) \\S_{1,8} &= (7\ 6\ 5\ 4\ 3\ 2\ 1) \\S_{2,7} &= (6\ 5\ 4\ 3\ 2) \\S_{3,6} &= (5\ 4\ 3)\end{aligned}$$

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Lemma

Fix $P_1, P_2 \in G$ and consider P'_1 and P'_2 . Then, P_1 can be changed to P_2 via legal moves if and only if P'_1 can be changed to P'_2 via legal moves.

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$$(i_k = 16 \ i_{k-1})(i_{k-1} \ i_{k-2}) \cdots (i_3 \ i_2)(i_2 \ i_1)(i_1 \ i_0 = 16)$$

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$L(K)$ is a group.

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- 2 Given a permutation in $L(K)$, performing the permutation in reverse order yields its inverse.
- 3 Permutation composition is an associative binary operation.
- 4 Any permutation in $L(K)$ begins and ends with a transposition including position 16, therefore compositions of elements of $L(K)$ will still be in $L(K)$.



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- $L(K) \leq A_n$

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Lemma

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$$\begin{aligned}
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 (2 \ 3 \ 4) &= (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7)^{-1}(3 \ 4 \ 5)(1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7) \\
 (4 \ 5 \ 6) &= (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7)^{-1}(5 \ 6 \ 7)(1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7) \\
 (5 \ 6 \ 7) &= (5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11)^{-2}(7 \ 8 \ 9)(5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11)^2 \\
 (6 \ 7 \ 8) &= (5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11)^{-1}(7 \ 8 \ 9)(5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11) \\
 (8 \ 9 \ 10) &= (5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11)^{-1}(9 \ 10 \ 11)(5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11) \\
 (9 \ 10 \ 11) &= (9 \ 10 \ 11 \ 12 \ 13 \ 14 \ 15)^{-2}(11 \ 12 \ 13)(9 \ 10 \ 11 \ 12 \ 13 \ 14 \ 15)^2 \\
 (10 \ 11 \ 12) &= (9 \ 10 \ 11 \ 12 \ 13 \ 14 \ 15)^{-1}(11 \ 12 \ 13)(9 \ 10 \ 11 \ 12 \ 13 \ 14 \ 15) \\
 (12 \ 13 \ 14) &= (9 \ 10 \ 11 \ 12 \ 13 \ 14 \ 15)(11 \ 12 \ 13)(9 \ 10 \ 11 \ 12 \ 13 \ 14 \ 15)^{-1} \\
 (13 \ 14 \ 15) &= (9 \ 10 \ 11 \ 12 \ 13 \ 14 \ 15)^2(11 \ 12 \ 13)(9 \ 10 \ 11 \ 12 \ 13 \ 14 \ 15)^{-2}
 \end{aligned}$$

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Theorem

A given configuration of the 15-puzzle can be changed legally into another configuration if and only if the standardized forms of the configurations can be transformed into each other by using an even permutation.

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- $|A_{15}| = \frac{15!}{2} = 653,837,184,000$

15-14 Configuration

Sam Lloyd's configuration:

<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>
<i>E</i>	<i>F</i>	<i>G</i>	<i>H</i>
<i>I</i>	<i>J</i>	<i>K</i>	<i>L</i>
<i>M</i>	<i>O</i>	<i>N</i>	

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Permutation required:

(14 15)

Reverse Configuration

Reverse order:

<i>O</i>	<i>N</i>	<i>M</i>	<i>L</i>
<i>K</i>	<i>J</i>	<i>I</i>	<i>H</i>
<i>G</i>	<i>F</i>	<i>E</i>	<i>D</i>
<i>C</i>	<i>B</i>	<i>A</i>	

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<i>C</i>	<i>B</i>	<i>A</i>	

Permutation required:

$$(15\ 4)(14\ 3)(13\ 2)(9\ 1)(10\ 5)(11\ 6)(12\ 7)$$

1-Blank Configuration

1-blank:

	<i>A</i>	<i>B</i>	<i>C</i>
<i>D</i>	<i>E</i>	<i>F</i>	<i>G</i>
<i>H</i>	<i>I</i>	<i>J</i>	<i>K</i>
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<i>M</i>	<i>N</i>	<i>O</i>	

Permutation required:

$$(12\ 9)(12\ 10)(11\ 10)(8\ 10)(2\ 1)(3\ 1)(4\ 1)$$