Legal Configurations of the 15-Puzzle

Andrew Chapple[1] Alfonso Croeze[1] Mhel Lazo[1] Hunter Merrill[2]

[1]Department of Mathematics
Louisiana State University
Baton Rouge, LA

[2]Department of Mathematics
Mississippi State University
Starkville, MS

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History

- Invented in the 1860s

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- Puzzle description

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Permutations

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Consider the set $A = \{1, 2, 3, 4, 5, 6\}$.

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Example

Consider the set $A = \{1, 2, 3, 4, 5, 6\}$. Then the permutation P,

$$P = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 5 & 2 & 3 & 6 \end{pmatrix}$$

changes 1 to 4, 2 to 1, 3 to 5, 4 to 2, 5 to 3, and *fixes*, or leaves unchanged, the element 6.

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Therefore, we can also write P as

$$P = (3\ 5)(1\ 4\ 2)$$

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- $(3\ 5)(3\ 5) = (3)(5) = I$

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- The set of all even permutations of n elements is denoted by A_n

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Lemma

The identity I, the permutation which fixes all elements, is even.

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\tau_1\tau_2\cdots\tau_s q_t^{-1}q_{t-1}^{-1}\cdots q_1^{-1} &= I \\
\tau_1\tau_2\cdots\tau_s q_t q_{t-1}\cdots q_1 &= I
\end{aligned}$$

The left hand side is a product of $s + t$ transpositions. Since Lemma 11 says that the identity on the right hand side is even, s and t must have the same parity.

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- The alternating group A_n under composition

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Theorem

$|O_n| = |A_n| = \frac{n!}{2}$.

Introduce Puzzle

Permutations performed on the puzzle are performed on positions and not on contents.

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4	3	2	1
5	6	7	8
12	11	10	9
13	14	15	16

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$\rightarrow$	$\rightarrow$	$\rightarrow$	$\uparrow$
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- The moves along this path are legal.

Special Moves

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Nine Moves

The following are all the legal moves not along the path:

$$
\begin{aligned}
S_{9,16} &= (15\ 14\ 13\ 12\ 11\ 10\ 9)\\
S_{10,15} &= (14\ 13\ 12\ 11\ 10)\\
S_{11,14} &= (13\ 12\ 11)\\
S_{7,10} &= (9\ 8\ 7)\\
S_{6,11} &= (10\ 9\ 8\ 7\ 6)\\
S_{5,12} &= (11\ 10\ 9\ 8\ 7\ 6\ 5)\\
S_{1,8} &= (7\ 6\ 5\ 4\ 3\ 2\ 1)\\
S_{2,7} &= (6\ 5\ 4\ 3\ 2)\\
S_{3,6} &= (5\ 4\ 3)
\end{aligned}
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Connections

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Lemma

Fix $P_1,\ P_2 \in G$ and consider P_1' and P_2'. Then, P_1 can be changed to P_2 via legal moves if and only if P_1' can be changed to P_2' via legal moves.

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$$(i_k = 16\ i_{k-1})(i_{k-1}\ i_{k-2}) \cdots (i_3\ i_2)(i_2\ i_1)(i_1\ i_0 = 16)$$

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Lemma

$L(K)$ is a group.

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2. Given a permutation in $L(K)$, performing the permutation in reverse order yields its inverse.
3. Permutation composition is an associative binary operation.
4. Any permuation in $L(K)$ begins and ends with a transposition including position 16, therefore compositions of elements of $L(K)$ will still be in $L(K)$.

□

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- $L(K) \leq A_n$

Results

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Every element of A_n can be written as a product of cycles of the form $(k\ k+1\ k+2)$.

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(1\ 2\ 3) &= (1\ 2\ 3\ 4\ 5\ 6\ 7)^{-2}(3\ 4\ 5)(1\ 2\ 3\ 4\ 5\ 6\ 7)^2 \\
(2\ 3\ 4) &= (1\ 2\ 3\ 4\ 5\ 6\ 7)^{-1}(3\ 4\ 5)(1\ 2\ 3\ 4\ 5\ 6\ 7) \\
(4\ 5\ 6) &= (1\ 2\ 3\ 4\ 5\ 6\ 7)^{-1}(5\ 6\ 7)(1\ 2\ 3\ 4\ 5\ 6\ 7) \\
(5\ 6\ 7) &= (5\ 6\ 7\ 8\ 9\ 10\ 11)^{-2}(7\ 8\ 9)(5\ 6\ 7\ 8\ 9\ 10\ 11)^2 \\
(6\ 7\ 8) &= (5\ 6\ 7\ 8\ 9\ 10\ 11)^{-1}(7\ 8\ 9)(5\ 6\ 7\ 8\ 9\ 10\ 11) \\
(8\ 9\ 10) &= (5\ 6\ 7\ 8\ 9\ 10\ 11)^{-1}(9\ 10\ 11)(5\ 6\ 7\ 8\ 9\ 10\ 11) \\
(9\ 10\ 11) &= (9\ 10\ 11\ 12\ 13\ 14\ 15)^{-2}(11\ 12\ 13)(9\ 10\ 11\ 12\ 13\ 14\ 15)^2 \\
(10\ 11\ 12) &= (9\ 10\ 11\ 12\ 13\ 14\ 15)^{-1}(11\ 12\ 13)(9\ 10\ 11\ 12\ 13\ 14\ 15) \\
(12\ 13\ 14) &= (9\ 10\ 11\ 12\ 13\ 14\ 15)(11\ 12\ 13)(9\ 10\ 11\ 12\ 13\ 14\ 15)^{-1} \\
(13\ 14\ 15) &= (9\ 10\ 11\ 12\ 13\ 14\ 15)^2(11\ 12\ 13)(9\ 10\ 11\ 12\ 13\ 14\ 15)^{-2}
\end{aligned}
$$

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- $|A_{15}| = \frac{15!}{2} = 653,837,184,000$

15-14 Configuration

Sam Lloyd's configuration:

A	B	C	D
E	F	G	H
I	J	K	L
M	O	N	

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Permutation required:

$$(14\ 15)$$

Reverse Configuration

Reverse order:

O	N	M	L
K	J	I	H
G	F	E	D
C	B	A	

Reverse Configuration

Reverse order:

O	N	M	L
K	J	I	H
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Permutation required:

$$(15\ 4)(14\ 3)(13\ 2)(9\ 1)(10\ 5)(11\ 6)(12\ 7)$$

1-Blank Configuration

1-blank:

	A	B	C
D	E	F	G
H	I	J	K
L	M	N	O

1-Blank Configuration

Standardize the configuration:

D	A	B	C
E	F	G	K
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$$(12\ 9)(12\ 10)(11\ 10)(8\ 10)(2\ 1)(3\ 1)(4\ 1)$$