AN ANALYSIS OF THE 15-PUZZLE

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Abstract. The 15-puzzle has been an object of great mathematical interest since its invention in the 1860s. The puzzle has 16 square slots on a square board. The first 15 slots have square pieces; the 16th slot is empty. The object of the puzzle is to slide orthogonally the square pieces into the empty spot, thus rearranging the pieces and changing which slot is vacant, until the desired configuration is obtained.

In 1878, famous puzzle maker Sam Lloyd swapped the 14th and 15th pieces and offered $1000 to the first person who could rearrange them to form the standard starting position. This problem contributed to the popularity of the puzzle as many attempted to solve it. In this paper, we simplify the work of Archer in his 1999 paper [**?**] and analyze the puzzle through permutations to determine which positions can be obtained from the standard starting position. We then evaluate specific positions, including Lloyd's famous challenge.

1. Basics

1.1. **Permutations.** A *permutation* of a set S is a bijection from S onto itself. If the set we are permuting is $A = \{1,\ 2, \ldots,\ n\}$, it is often convenient to represent a permutation σ as follows:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \sigma(1) & \sigma(2) & \sigma(3) & \cdots & \sigma(n) \end{pmatrix}.$$

For instance, consider the set $A = \{1, 2, 3, 4, 5, 6\}$. Then the permutation π,

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 5 & 2 & 3 & 6 \end{pmatrix},$$

sends 1 to 4, 2 to 1, 3 to 5, 4 to 2, 5 to 3, and *fixes*, or leaves unchanged, the element 6.

The permutation above can be more compactly written in *cycle notation*. We note that 1 becomes 4, which becomes 2, which becomes

1; we express this as the *cycle* (1 4 2) (cycles are always read left to right). Similarly, 3 becomes 5 which becomes 3, so we write that as (3 5). The element 6 does not change, so that part of the permutation is written simply as (6). Altogether, the cyclic description of π is

$$\pi = (1\ 4\ 2)(3\ 5)(6).$$

However, since the cycle (6) does not actually change anything, it may be omitted:

$$\pi = (1\ 4\ 2)(3\ 5)$$

In this example the two cycles which compose π are *disjoint*, having no common elements, and so they commute. Therefore, we can also write π as

$$\pi = (3\ 5)(1\ 4\ 2).$$

In this paper, when nondisjoint cycles are to be performed in succession, the rightmost cycle is performed first. Cycles which consist of only two elements, such as the cycle (3 5) in π, are referred to as 2-cycles, or more commonly, as *transpositions*. The transposition (3 5) can also be written as (5 3), as both have the effect of swapping the elements 3 and 5. Subsequently, any transposition is its own inverse.

Theorem 1.1. *Every cycle may be written as a product of transpositions.*

Proof. Consider a cycle C,

$$C = (x_1\ x_2\ x_3\ x_4\ \ldots\ x_{k-2}\ x_{k-1}\ x_k)$$

where x_i is an element of the set being permuted. Then C may be expressed as

$$C = (x_1\ x_k)(x_1\ x_{k-1}) \ldots (x_1\ x_4)(x_1\ x_3)(x_1\ x_2).$$

To see that this works, note that x_1 becomes x_2 by the rightmost transposition. For all other x_i, $i < k$, x_i will become x_1 by the transposition $(x_1\ x_i)$, and it will subsequently become x_{i+1} by the next transposition, $(x_1\ x_{i+1})$. □

Refering back to our previous example, we consider the number of possible permutations on the set A. Note that we have 6 choices for the first element to which we want to assign an output, the numbers 1 to 6. The second element we examine has 5 possible outputs, since we already assigned one of the numbers 1 to 6 to the first element and cannot repeat them. Continuing in this manner, there are thus $6 * 5 * 4 * 3 * 2 * 1 = 6! = 720$ permutations of the set A. Using a similar method, it can be easily demonstrated that, given a set with n elements, there are $n!$ permutations of that set.

The set of all permutations of a set $S = \{1, 2, \ldots, n\}$ is called the *symmetric group* on n letters, and is denoted S_n. We showed in the previous paragraph that

$$|S_n| = n!$$

Now, let us once again refer to our permutation π. Using the technique of Theorem **??**, we write π as

$$\pi = (1\ 4\ 2)(3\ 5) = (1\ 2)(1\ 4)(3\ 5).$$

Note that we used an odd number of transpositions to express π, and thus call π an *odd* permutation. Any permutation which may be written as the product of an odd number of transpositions is called odd; otherwise, we call it *even*.

Sometimes, when working with S_n, we only want to consider permutations of one parity. The set of odd permutations in S_n is denoted by O_n, and the *alternating group*, or the set of even permutations in S_n, is denoted by A_n.

A few more theorems are necessary.

Theorem 1.2. *The identity I, the permutation which fixes all elements, is even.*

(Grove, p.17). To show that the identity is even, we need to show that it is not possible to write it as the product of an odd number of transpositions. Suppose, to the contrary, that we have done so, $1 = (ab)\cdots$, using a minimal number of transpositions and with the smallest possible number of a's appearing. At least one more a must appear, since $1a \neq b$, so suppose (ac) is the next one to the right. Note that $(de)(ac) = (ac)(de)$ if (de) and (ac) are disjoint and $(dc)(ac) = (ad)(cd)$, so we may move the second a to the left and write $1 = (ab)(af)\cdots$, with the same minimality conditions met. But now if $b = f$ we may reduce the number of transpositions by 2, and if $b \neq f$, then $(ab)(af) = (af)(bf)$ and we may reduce the number of a's, in both cases a contradiction. □

Theorem 1.3. *If $\sigma \in S_n$, then σ may be written as the product of an even number of transpositions if and only if σ can not be written as the product of an odd number of transpositions.*

Proof. Suppose σ can be written as a product of transpositions in two ways. Then we may write:

$$\sigma = \tau_1\tau_2 \cdots \tau_s = q_1q_2 \cdots q_t$$

Multiply both sides by the inverse of the right hand side,

$$\begin{aligned}
\tau_1\tau_2 \cdots \tau_s(q_1q_2 \cdots q_t)^{-1} &= q_1q_2 \cdots q_t(q_1q_2 \cdots q_t)^{-1} \\
\tau_1\tau_2 \cdots \tau_s(q_1q_2 \cdots q_t)^{-1} &= I \\
\tau_1\tau_2 \cdots \tau_s q_t^{-1}q_{t-1}^{-1} \cdots q_1^{-1} &= I \\
\tau_1\tau_2 \cdots \tau_s q_t q_{t-1} \cdots q_1 &= I
\end{aligned}$$

The left hand side is a product of $s+t$ transpositions. Since Theorem **??** showed that the identity on the right hand side is even, s and t must have the same parity.

□

Theorem 1.4. $|O_n| = |A_n| = \frac{n!}{2}$.

Proof. Fix $\sigma \in S_n$. Let $f(\sigma) = (1\ 2)\sigma$ Note that

$$f(\sigma) \in \begin{cases} A_n & \text{if } \sigma \in O_n \\ O_n & \text{if } \sigma \in A_n. \end{cases}$$

Consider any $\sigma_1,\ \sigma_2 \in S_n$. If $f(\sigma_1) = f(\sigma_2)$, then

$$\begin{aligned}
(1\ 2)\sigma_1 &= (1\ 2)\sigma_2 \\
(1\ 2)(1\ 2)\sigma_1 &= (1\ 2)(1\ 2)\sigma_2 \\
\sigma_1 &= \sigma_2
\end{aligned}$$

Therefore, f is *injective.*
Now consider any $\sigma_2 \in S_n$. Then since

$$f((1\ 2)\sigma_2) = (1\ 2)(1\ 2)\sigma_2 = \sigma_2,$$

we have that f is surjective.

Since f is injective and surjective, f is *bijective.*
This means that every $\sigma_1 \in A_n$ can be mapped to exactly one $\sigma_2 \in O_n$. Likewise, every element of O_n is mapped to exactly one element of A_n. Therefore,

$$|O_n| = |A_n|$$

By Theorem **??**, every element in S_n belongs to exactly one of A_n or O_n. So,

$$|O_n| + |A_n| = |S_n|$$
$$2|A_n| = |S_n|$$
$$|A_n| = \frac{|S_n|}{2} = \frac{n!}{2}$$

□

1.2. **Groups.** A *group* is a combination of a set S and a binary operation $*$ that has the following properties:

(1) The set is non-empty.
(2) The set is closed under the operation; the result of the operation on any two elements of the set must be another element in the set.
(3) The set is associative: $(a * b) * c = a * (b * c)$.
(4) The set contains an identity (denoted by I, Id or e) that fixes any element $\sigma \in S$.
(5) The set must contain an inverse for each element in the set; for any $\sigma \in S$, there exists another element $\sigma' \in S$ such that $\sigma * \sigma' = \sigma' * \sigma = I$.

Examples of groups include the integers under addition, S_n, and A_n.

A subset H of a group G is called a *subgroup* of G if the operation on G restricts to a binary operation on H under which H is itself a group.

2. The Puzzle

Before an analysis of the 15-puzzle can be accomplished, we must specify that the permutations of the puzzle are performed on the positions of the puzzle, rather than the contents of the positions. That is, a transposition (1 2) will swap the contents of positions 1 and 2, rather than swapping contents labeled 1 and 2. The positions of our 15-puzzle will be labeled like this:

4	3	2	1
5	6	7	8
12	11	10	9
13	14	15	16

This labeling follows a specific path from position 1 to the position 16 that will be called the "snake" path for the remainder of this paper.

The contents of the positions of our 15-puzzle will be labeled using letters to avoid some confusion (note that, as we are permutating positions, the actual labels of the contents do not matter):

A	B	C	D
E	F	G	H
I	J	K	L
M	N	O	

This configuration will be called the "initial" configuration for the remainder of the paper. As an example, if one wanted to swap F and J, the transposition required to do so is (6 11).

Define the following set:

$$G = \{\text{all possible configurations, with the blank space anywhere}\}$$

For any configuration $P \in G$, define P' to be the configuration where the blank in P has been snaked to position 16. In some cases, $P = P'$. We will call P' the *standardization* of P.

Lemma 2.1. *Fix $P_1,\ P_2 \in G$ and consider P_1' and P_2'. P_1 can be changed to P_2 via legal moves if and only if P_1' can be changed to P_2' via legal moves.*

Proof. First, assume P_1' can be changed to P_2'. Then, we can change P_1 to P_2 by changing P_1 to P_1', P_1' to P_2', and then P_2' to P_2. Similarly, if we assume that the change from P_1 to P_2 is possible, then we can change P_1' to P_2' by changing P_1' to P_1, P_1 to P_2, and then P_2' to P_2. □

This lemma may seem trivial, but it in fact greatly reduces the number of board configurations which need to be evaluated as we may now consider only configurations in G for which the blank is in position 16. Denote by $K \subset G$ all configurations of the board for which the blank is in position 16. We note that legal moves consist of transposing the position the blank occupies with one of the orthogonally adjacent positions, which we call *neighbors*. Since we are considering only configurations in K, our blank must start and end at position 16. So our legal moves, denoted $L(K)$ are permutations of the form:

$$(i_k = 16\ i_{k-1})(i_{k-1}\ i_{k-2})\cdots(i_3\ i_2)(i_2\ i_1)(i_1\ i_0 = 16)$$

where i_s is a neighbor of i_{s+1} for $0 \leq s < k$

Lemma 2.2. *$L(K)$ is a group.*

Proof. We will show that $L(K)$ fulfills all the properties of a group:

(1) $L(K)$ is nonempty. As an example we have (16 15)(15 16), which also happens to be the identity.
(2) Given a permutation in $L(K)$, performing the permutation in reverse order yields its inverse.
(3) Permutation composition is an associative binary operation.
(4) Any permuation in $L(K)$ begins and ends with a transposition including position 16, therefore compositions of elements of $L(K)$ will still be in $L(K)$.

□

Lemma 2.3. $L(K) \leq A_{15}$.

Proof. Going from one configuration to another, we had to move the blank space around, but we always returned it to 16. If we moved it left, we always had to move it back right; if we moved it up, we always had to move it back down. As a result, we always used an *even number* of swaps to change configurations; therefore, we used an even number of transpositions. This tells us that no odd permutation could ever describe a configuration legally obtained from the initial configuration. Of all the permutations in S_{15}, only those in A_{15} are potentially possible. Therefore, $L(K) \subseteq A_{15}$. Since we have already shown that $L(K)$ is a group, it immediately follows that $L(K) \leq A_{15}$. □

There are legal moves, associated with nine transpositions, which swap two tiles not along the snaked path.

For example, we want to determine how the move associated with (11 14) changes the board. Starting with the blank at 16, we move it along the snake until we reach the first position used in the transposition (in this case, 14), perform the transposition, then move the blank along the path back to 16; the blank's path is 16, 15, 14, 11, 12, 13, 14, 15, 16. Continuing in this way, we get the permutation:

$$(16\ 15)(15\ 14)(14\ 13)(13\ 12)(12\ 11)(11\ 14)(14\ 15)(15\ 16) = (13\ 12\ 11)$$

This permutation is clearly an element of $L(K)$, and we denote it by $S_{11,14}$. The others are listed below.

$$\begin{aligned}
S_{9,16} &= (15\ 14\ 13\ 12\ 11\ 10\ 9)\\
S_{10,15} &= (14\ 13\ 12\ 11\ 10)\\
S_{11,14} &= (13\ 12\ 11)\\
S_{7,10} &= (9\ 8\ 7)\\
S_{6,11} &= (10\ 9\ 8\ 7\ 6)\\
S_{5,12} &= (11\ 10\ 9\ 8\ 7\ 6\ 5)\\
S_{1,8} &= (7\ 6\ 5\ 4\ 3\ 2\ 1)\\
S_{2,7} &= (6\ 5\ 4\ 3\ 2)\\
S_{3,6} &= (5\ 4\ 3)
\end{aligned}$$

Note that

$$\begin{aligned}
S_{9,16}^{-1} &= (9\ 10\ 11\ 12\ 13\ 14\ 15)\\
S_{10,15}^{-1} &= (10\ 11\ 12\ 13\ 14)\\
&\vdots\\
S_{3,6}^{-1} &= (3\ 4\ 5)
\end{aligned}$$

3. The Result

When moving the tiles around the board, we were working in S_{16} since there are 16 total squares whose contents we were changing. However, because we only considered configurations where the blank was in 16, we never changed the contents of 16 in going from one configuration to another; since we effectively fixed 16, we were really working in S_{15}. By Lemma **??**, we showed that only even permutations, i.e. the permutations in A_{15}, are possible. But which ones?

The following two theorems will help address this question.

Theorem 3.1. *Every element of S_n may be written as a product of transpositions of the form $(k\ k+1)$.*

Proof. Choose $\sigma \in S_n$ and select transpositions $\tau_1, \ldots \tau_k$ (possibly by using the technique shown in Theorem **??**) such that

$$\sigma = \tau_1 \tau_2 \tau_3 \ldots \tau_k$$

Every τ_i can be written in the following way:

For each τ_i, we may write for distinct $k, l \in 1, 2, \ldots, n$ $\tau_i = (k\ l)$. Since $(k\ l) = (l\ k)$, we assume WLOG that $k < l$. Then,

$$\tau_i = (k\ l)$$

which we may then expand to get

$$\tau_i = (k\ k{+}1)(k{+}1\ k{+}2)\cdots(l{-}1\ l)(l{-}2\ l{-}1)\cdots(k{+}1\ k{+}2)(k\ k{+}1)$$

Repeating this for $i = 1,\ 2, \ldots, k$ we get σ as a product of adjacent transpositions. □

Theorem 3.2. *Every element of A_n can be written as a product of cycles of the form $(k\ k+1\ k+2)$.*

Proof. From Theorem **??**, we found that every permutation can be written as a product of adjacent transpositions of the form

$$(k\ k+1)(k+1\ k+2)\cdots(l-1\ l)(l-2\ l-1)\cdots(k+1\ k+2)(k\ k+1)$$

Focusing on two "side-by-side" adjacent transpositions at a time, we find the following three cases:

(1) $(k\ k+1)(k+1\ k+2)$
(2) $(k+1\ k+2)(k\ k+1)$
(3) $(a\ a+1)(b\ b+1)$ where $\{a, a+1\} \cap \{b, b+1\} = \emptyset$

We may rewrite the first two cases as follows:

(1) $(k\ k+1)(k+1\ k+2) = (k\ k+1\ k+2)$
(2) $(k+1\ k+2)(k\ k+1) = (k\ k+2\ k+1)$

For the third case, we assume WLOG (since disjoint permutations commute) that $a < b$. Then we may rewrite $(a\ a+1)(b\ b+1)$ as

$$(a\ a+1)(a+1\ a+2)(a+1\ a+2)\cdots(b-1\ b)(b-1\ b)(b\ b+1).$$

Since $(a\ a+1)(b\ b+1)$ is an even permutation, the permutation above on the right side will have an even number of transpositions. We may then use the first case to rewrite the transpositions as needed.

Since elements of A_n use an even number of adjacent transpositions, no transposition is omitted when we combine them in such a way. Therefore, every element of A_n can be written as a product of cycles of the forms

$$(k\ k+1\ k+2) \quad \text{or} \quad (k\ k+2\ k+1).$$

However, since

$$(k\ k+1\ k+2) = (k\ k+2\ k+1)^2$$

all elements can actually be written using only the form

$$(k\ k+1\ k+2).$$

□

So, we wish to show that all cycles of the form $(k\ k+1\ k+2)$ can be formed using the nine vertical transpositions presented earlier. Note that we already have (3 4 5), (7 8 9), and (11 12 13) as inverses of cycles formed from three of the transpositions. The other necessary cycles are listed here.

$$\begin{aligned}
(1\,2\,3) &= (1\,2\,3\,4\,5\,6\,7)^{-2}(3\,4\,5)(1\,2\,3\,4\,5\,6\,7)^{2}\\
(2\,3\,4) &= (1\,2\,3\,4\,5\,6\,7)^{-1}(3\,4\,5)(1\,2\,3\,4\,5\,6\,7)\\
(4\,5\,6) &= (1\,2\,3\,4\,5\,6\,7)^{-1}(5\,6\,7)(1\,2\,3\,4\,5\,6\,7)\\
(5\,6\,7) &= (5\,6\,7\,8\,9\,10\,11)^{-2}(7\,8\,9)(5\,6\,7\,8\,9\,10\,11)^{2}\\
(6\,7\,8) &= (5\,6\,7\,8\,9\,10\,11)^{-1}(7\,8\,9)(5\,6\,7\,8\,9\,10\,11)\\
(8\,9\,10) &= (5\,6\,7\,8\,9\,10\,11)^{-1}(9\,10\,11)(5\,6\,7\,8\,9\,10\,11)\\
(9\,10\,11) &= (9\,10\,11\,12\,13\,14\,15)^{-2}(11\,12\,13)(9\,10\,11\,12\,13\,14\,15)^{2}\\
(10\,11\,12) &= (9\,10\,11\,12\,13\,14\,15)^{-1}(11\,12\,13)(9\,10\,11\,12\,13\,14\,15)\\
(12\,13\,14) &= (9\,10\,11\,12\,13\,14\,15)(11\,12\,13)(9\,10\,11\,12\,13\,14\,15)^{-1}\\
(13\,14\,15) &= (9\,10\,11\,12\,13\,14\,15)^{2}(11\,12\,13)(9\,10\,11\,12\,13\,14\,15)^{-2}
\end{aligned}$$

We have shown that all permutations of the form $(k\ k+1\ k+2)$ are possible, and since we know from Theorem **??** that all permutations in A_n can be generated from those permutations, we have in fact shown that all permutations in A_{15} can be legally obtained: that is, $L(k) = A_{15}$.

However, the initial configuration does not have to be the "normal" starting configuration we included earlier; it could be any standard configuration. But, by Lemma **??**, we know that *any* configuration can be changed to an equivalent standard configuration. We can therefore state our result as follows.

Theorem 3.3. *A given configuration of the 15-puzzle can be changed legally into another configuration if and only if the standardardized forms of the configurations can be transformed into each other by an even permutation.*

Thus, by Theorem **??**, there are $\frac{15!}{2} = 653,837,184,000$ legal standard configurations.

4. Applications

One configuration worth mentioning is the configuration in which the contents of the positions are placed in reverse order. In keeping with the definition above in which the positions are labeled in consecutive order along the "snake" and the contents of each position are labeled in consecutive order following the traditional left-to-right and top-to-bottom scheme, the change from the initial position to the reverse-order configuration would look like this:

O	N	M	L
K	J	I	H
G	F	E	D
C	B	A	

To attain this configuration, the contents of the 4 and 15 positions must be swapped, the 3 and 14, etc. This move can be written as a product of transpositions:

$$(15\ 4)(14\ 3)(13\ 2)(9\ 1)(10\ 5)(11\ 6)(12\ 7)$$

This configuration requires an odd number of transpositions and is thus impossible to obtain from the starting configuration using only legal moves. Conversely, no legal configurations are attainable from this configuration using only legal moves.

Now let us examine Sam Lloyd's configuration:

A	B	C	D
E	F	G	H
I	J	K	L
M	O	N	

Only the N and O have been moved from their initial positions, so we can describe this configuration, relative to the initial configuration, with a single transposition: (14 15). But, since we used an odd number of transpositions, we immediately conclude that this configuration could not have been obtained legally from the normal initial configuration. One should always be wary of a mathematician's challenge.

Another configuration of interest is the 1-blank, in which the blank is in the top left corner and the contents of every position are shifted to the right one place:

	A	B	C
D	E	F	G
H	I	J	K
L	M	N	O

Standardize the configuration:

D	A	B	C
E	F	G	K
L	H	I	J
M	N	O	

Is this configuration possible? To achieve this configuration from the initial configuration, one must use the permutation

$$(12\ 9)(12\ 10)(11\ 10)(8\ 10)(2\ 1)(3\ 1)(4\ 1)$$

This permutation is odd, and thus the configuration is impossible to obtain through legal moves.

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