A Classification of Frieze Patterns

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Outline

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Important Definitions

Definition

A group G is any non-empty set with a binary operation that has an identity element, every element in the group has an inverse, it is closed under the binary operation, and it is associative

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An isometry is a transformation on the plane which preserves distances and is bijective

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Isometries of the Complex Plane

- Every isometry on the complex plane follows one of two forms...
  1. $f(z) = \alpha z + \beta$ or
  2. $f(z) = \alpha \bar{z} + \beta$
- Where $|\alpha| = 1$ and $\alpha, \beta \in \mathbb{C}$

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Isometries and Groups

- The isometries of some figure $F \subseteq \mathbb{C}$ that fix F form a group
- $I(F) = \{g \in I(\mathbb{C}) : g(F) = F\}$
- Any two isometries of F multiplied will still give you F
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Standard Frieze Group

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A frieze group is any group G of isometries in the complex plane such that for every $g \in G$, $g(\mathbb{R}) = \mathbb{R}$ and the translations in the group form an infinite cyclic group generated by τ where $\tau(z) = z + 1$

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Important Proof

- We can apply isometries of the complex plane, to frieze groups, with even more precision

Proposition

For any isometry of a frieze group, $\alpha =$1 or -1 and $\beta \in \mathbb{R}$

Proof.

First, observe $f(0) = \alpha(0) + \beta = \beta$ which implies $\beta \in \mathbb{R}$ because $f(0) \in \mathbb{R}$. Next, observe $f(1) = \alpha(1) + \beta$. Since both $\beta, f(1) \in \mathbb{R}$, we know that $\alpha \in \mathbb{R}$. We have already established $|\alpha| = 1$, thus α = 1 or -1 □

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Isometries of Frieze Groups

- Using the equation for an isometry of a frieze group, we find that there are five different types of isometries of G.
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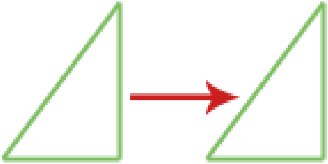
If $\alpha = 1$

- $f(z) = \alpha z + \beta$: Then $z + \beta$. This is an element of T, the translations, so we kno β must equal $m \in \mathbb{Z}$



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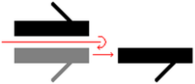
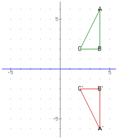
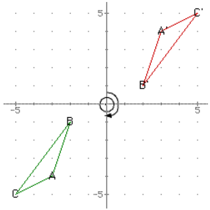
- $f(z) = \alpha\bar{z} + \beta$ Then $f(z) = \bar{z} + \beta$. If $\beta = 0$ and $f(z) = \bar{z}$, f will be a reflection about the x-axis. If $\beta = m \in \mathbb{Z}$ then f will be a reflection about the x-axis and then a translation by an integer m. By squaring f we find out that β can also equal $m + \frac{1}{2}$. This will be a glide reflection.





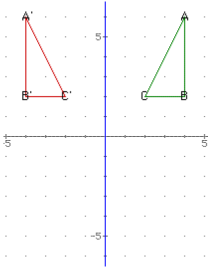
If $\alpha = -1$

- $f(z) = \alpha z + \beta$: Then $f(z) = -z + \beta$. This is a 180° rotation.



If $\alpha = -1$

- $f(z) = \alpha\bar{z} + \beta$: Then $f = -\bar{z} + \beta$. This is a vertical reflection.



Normal Subgroups

Definition

If H is a subgroup of G, we say H is a normal subgroup of G if for all $x \in G$, $x^{-1}Hx \subseteq H$

- A normal subgroup H of a group G is denoted $H \triangleleft G$
- The set of all translations T is a normal subgroup of any frieze group G

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If H is a subgroup of G and $x, y \in G$, then x and y are congruent mod H if $y^{-1}x \in H$

- In order for any two isometries f and g to be congruent mod T, they must be of the same form
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General Quotient Group

Definition

For $H \triangleleft G$, we denote the set of cosets of H as the quotient group G/H, which is equal to $\{gH \mid g \in G\}$ together with an operator given by $gH \bullet fH = gfH$ where $g, f \in G$

- For any group of isometries G, the order of G/T must be less than or equal to five, because there are only five different types of isometries

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LaGrange's Theorem Applied

- We can apply LaGrange's theorem which states that for any finite group G, the order of any subgroup H of G must divide the order of G
- Again defining G to be a frieze group, we now know that the order of G/T must be either one, or an even number
- Hence, for any frieze group G, the order of G/T must be either one, two, or four

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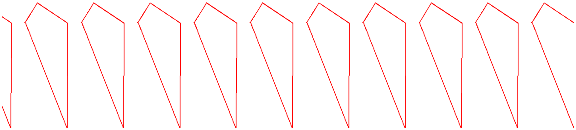
Types of Frieze Patterns

- Because of LaGranges Theorem, we know that the order of G/T must be either one, two, or four. We also know that G/T must contain T.

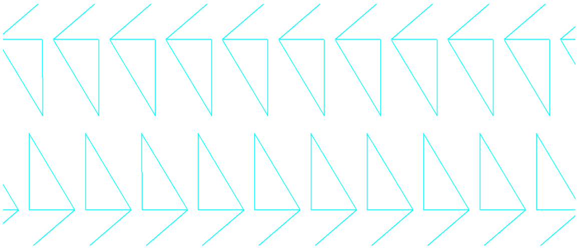
Examples, Order 1

- $G/T = < T >$ This group is just translations.



Examples, Order 2

- Because T, the translations, is included in each of the orders of G/T, there are only four possibilites for groups of order two.
- $< T, \rho T >$ This group consists of 180^o rotations.



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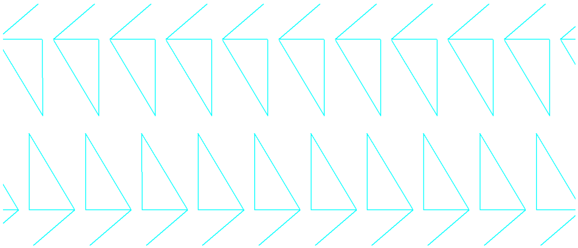
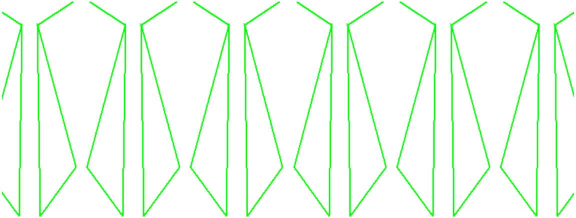


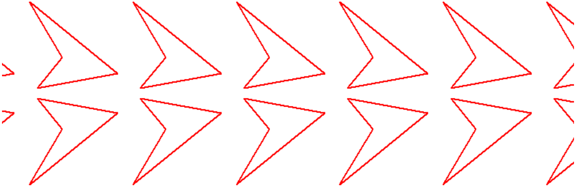
Example, Order 2

- $< T, vT >$ This group consists of vertical reflections.



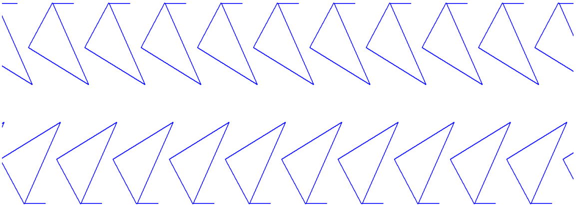
Examples, Order 2

- $< T, hT >$ This group consists of horizontal reflections.



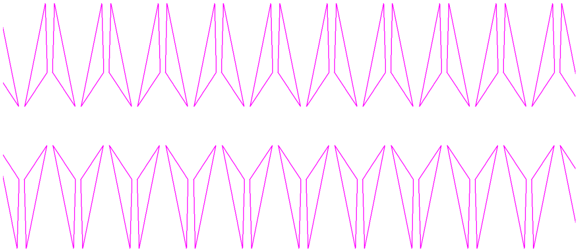
Example, Order 2

- $< T, gT >$ This group consists of glide reflections.



Examples, Order 4

- Because T must be included in each group, and because g and h cannot be included together in the same group, there are only two possible groups of order four.
- $< T, vT, \rho T, gT >$ This group consists of vertical reflections, rotations, and glide reflections



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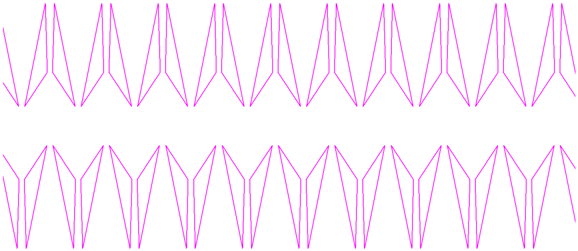

Examples, Order 4

- $< T, vT, \rho T, hT >$ This group consists of vertical and horizontal reflections, and rotations