A CLASSIFICATION OF FRIEZE PATTERNS

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Abstract. This paper explores the properties of frieze patterns. We will begin by displaying background information such as the definition of a group and of an isometry. Then, by showing that any frieze pattern is a group, we apply properties of isometries to the frieze patterns. This allows us to define five types of isometries and explore their relationship to each other, and the pattern as a whole. In exploring their properties, we classify every freize pattern as one of seven types.

1. Introduction

A frieze pattern is a patterned band of repeated design. Frieze patterns are often seen as border patterns found on architecture, pottery, stitching, and wall paper. A frieze pattern will always have some type of symmetry. Different types of frieze patterns can be found from the different symmetries they possess. This paper will analyze the mathematics behind the different symmetries of frieze patterns, and why only specific symmetries can make a frieze pattern. In this paper frieze patterns will be looked at as infinite strips of repeating symmetries. This analysis of frieze patterns will begin with two very important definitions.

2. Important Definitions

Definition 2.1. A Group is a non-empty set together with an operation that is closed under that operation, associative, has an identity element, and every element in the set has an inverse.

Definition 2.2. An isometry is a transformation of the plane which preserves distances and is bijective.

3. Isometries of a Figure Form a Group

We find that the isometries of some figure $F \subseteq \mathbb{C}$ that fix F form a group. Using the definition of a set we see that $I(F) = \{g \in I(\mathbb{C}) : g(F) = F\}$. So any isometry $g \in \mathbb{C}$ is an element of the isometries of the complex plain, and maps F to itself. So any two isometries composed will always give you an element of F. This proves that $I(F)$ is closed. The "do nothing" isometry acts as the identity, the isometries of F are associative, and the inverse of an isometry exists because the inverse of an isometry is just another isometry. So we see that $I(F)$ is a group.

4. Isometries of a Group in the Complex Plane

The isometries of a group are represented in the complex plane when $\beta, \alpha \in \mathbb{C}$ and $|\alpha| = 1$.

Lemma 4.1. *Let G be an isometry of $\mathbb{C}$ with $G(0) = 0, G(1) = 1$*
If:
(1) $G(i) = i$, then $G(z) = z$ for all $z \in \mathbb{C}$
(2) $G(i) = -i$, then $G(z) = \bar{z}$ for all $z \in \mathbb{C}$

Proof. Case 1:

$$G(0) = 0, G(1) = 1, G(i) = i$$

Fix

$$z = x + iy \quad \text{with} \quad x, y \in \mathbb{R}$$

Set

$$G(z) = v + iw \quad \text{with} \quad v, w \in \mathbb{R}$$

(1) We have that $\text{dist}[G(z), 0] = \text{dist}[z, 0]$, and $\text{dist}[(v, w), (0, 0)] = \text{dist}[(x, y), (0, 0)]$ so $v^2 + w^2 = x^2 + y^2$

(2) We have that $\text{dist}[G(z), 1] = \text{dist }[z, 1]$ and $\text{dist}[(v, w), (1, 0)] = \text{dist}[(x, y), (1, 0)]$ so $(w-1)^2+v^2 = (x-1)^2+y^2$ which expands to $w^2+v^2-2w+1 = x^2 + y^2 - 2y + 1$ which simplifies to $w = y$

(3) We have that $\text{dist}[G(z), G(1)] = \text{dist}[z, 1]$ and $\text{dist}[(v, w), (0, 1)] = \text{dist}[(x, y), (0, 1)]$ so $v^2+(w-1)^2 = x^2+(y-1)^2$ which expands to $v^2+w^2-2w+1 = x^2 + y^2 - 2y + 1$

which simplifies to $w = y$

Thus, $G(z) = v + iw = x + iy = z$

Case 2:

$$G(0) = 0, G(1) = 1, G(i) = -i$$

Define $\bar{G}(z) = \overline{G(z)}$ which sends

(1) 0 to $\bar{G}(0) = 0$

(2) 1 to $\bar{G}(1) = 1$

(3) i to $\bar{G}(i) = \overline{G(i)} = \bar{-i} = i$

Moreover, $\bar{G} = - \circ G$ is an isometry
$\bar{G}$ must then be the identity
Therefore, $\bar{G}(z) = z$ for all z So $G(z) = \bar{z}$

□

Theorem 4.2. *Fix $\alpha, \beta \in \mathbb{C}$ with $|\alpha| = 1$*

Then the maps

(1) $z \mapsto \alpha z + \beta$

(2) $z \mapsto \alpha \bar{z} + \beta$

are isometries of $\mathbb{C}$ and every isometry of $\mathbb{C}$ has one of these forms.

Proof. Let F be a fixed isometry of $\mathbb{C}$

$$\begin{aligned} \alpha &= F(1) - F(0) \\ \beta &= F(0) \end{aligned}$$

From 4.1,

$$G(z) = \frac{F(z) - F(0)}{F(1) - F(0)} = \frac{F(z) - \beta}{\alpha} = \begin{cases} \alpha z \\ \alpha \bar{z} \end{cases}$$

Multiplying by α gives us

$$G(z) = F(z) - \beta = \begin{cases} \alpha z \\ \alpha \bar{z} \end{cases}$$

Adding β, we can see the only two possible forms of the isometry $F(z)$.

$$F(z) = \begin{cases} \alpha z + \beta \\ \alpha \bar{z} + \beta \end{cases}$$

□

5. Standard Frieze Group G

5.1. **Frieze Group.** - A frieze group is any group G of isometries in the complex plane such that for every $g \in G$, $g(\mathbb{R}) = \mathbb{R}$ and the translations in the group form an infinite cyclic group generated by τ where $\tau(z) = z + 1$

Infinite Cyclic Group - A group is said to be infinitly cyclic if every $g \in G$, is equal to τ^m for some $m \in \mathbb{Z}$ where it is possible to generate infinitly many elements and τ^M is distinct.

Proposition 5.1. *For $f \in G$, where $f(z)$ either equals $\alpha z + \beta$ or $\alpha \bar{z} + \beta$, we have $\alpha = 1$ or $\alpha = -1$ and $\beta \in \mathbb{R}$.*

Proof. Case 1: $f(z) = \alpha z + \beta$

Then, $f(0) = \alpha(0) + \beta = \beta$ which implies $\beta \in \mathbb{R}$. This holds true because $0 \in \mathbb{R}$ and by definition of a frieze group, $f(0) \in \mathbb{R}$. Now observe $f(1) = \alpha z + \beta = \alpha(1) + \beta = \alpha + \beta$

We have shown that $\beta \in \mathbb{R}$ and we know $f(1) \in \mathbb{R}$ by definition of a frieze group. Therefore, $f(1) - \beta = \alpha \in \mathbb{R}$ which tells us $\alpha = 1$ or $\alpha = -1$ because we know from Theorem 3.2 that $|\alpha| = 1$

Case 2: $f(z) = \alpha \bar{z} + \beta$

Using the same argument to Case one, we see for $f(z) = \alpha \bar{z} + \beta$, α must be 1 or -1 and $\beta \in \mathbb{R}$

Thus, for all cases of $f(z)$, α equals 1 or -1 and $\beta \in \mathbb{R}$ □

6. Isometries of a Frieze Group

As we have seen there are two cases we must consider for the isometries of a frieze group. For any element f of the frieze group G, $f(z) = \alpha z + \beta$ or $f(z) = \alpha \bar{z} + \beta$, and because $\alpha = \pm 1$ we have two such scenerios for each case.

6.1. **Case 1.**

6.1.1. $\alpha = 1$. Then $f(z) = z + \beta$ which is an element of T, so β must be equal to some $m \in \mathbb{Z}$

6.1.2. $\alpha = -1$. Then $f(z) = -z + \beta$. This is a 180^o rotation.

6.2. **Case 2.**

6.2.1. $\alpha = 1$. Then $f(z) = \bar{z} + \beta$. This is a reflection about the x-axis. If $\beta = 0$ then the equation will be $f(z) = \bar{z}$, which is just a horizontal reflection. Now, if we take f^2 we know that we will still get an isometry. So,

$$f(f(z)) = \overline{(\bar{z} + \beta)} + \beta = z + \bar{\beta} + \beta = z + \beta + \beta = z + 2\beta$$

We find that f^2 is a translation. Because f^2 is a translation, $2\beta \in \mathbb{Z}$, so we get that β is equal to half of an integer. This means that for any $m \in \mathbb{Z}$, $\beta = m$, or $\beta = \frac{1}{2} + m$. If $\beta = m$ then f is a horizontal reflection then a translation by m. If $\beta = \frac{1}{2} + m$ then f is a glide reflection.

6.2.2. $\alpha = -1$. Then $f = -\bar{z} + \beta$. This will be a vertical reflection.

So we see that there are five possible types of isometries for any group G

7. Normal Groups

If H is a subgroup of G, we say H is normal in G if for all $x \in G$, $x^{-1}Hx \subseteq H$. In every scenerio, $H \subseteq x^{-1}Hx$, so if H is a normal subgroup of G, $H \subseteq x^{-1}Hx \subseteq H$ which implies $x^{-1}Hx = H$. From here on we use $H \lhd G$ to denote that H is a normal subgroup of G. For $H \lhd G$, we denote the set of cosets of H as "G/H"

Proposition 7.1. *The set of translations T is normal in a frieze group G*

Proof. Let $\tau \in T$ so that $\tau(z) = z + 1$ and $g \in G$. If T is normal in G, then $g^{-1}Tg \subseteq T$ for all $g \in G$. We choose $\tau^m \in T$ so that $m \in \mathbb{Z}$. Then, we first pick an arbitratry g of the form $g(z) = \alpha z + \beta$. Then $g^{-1}(z) = \frac{z-\beta}{\alpha}$. Then, $(g^{-1} \circ \tau^m \circ g)(z) = g^{-1} \circ ((\tau^m \circ g)(z)) =$

$g^{-1}(\tau^m(g(z))) = g^{-1}((\alpha z+\beta)+m) = \frac{((\alpha z+\beta)+m)-\beta}{\alpha} = z + \frac{m}{\alpha}$. Next, we pick an arbitratry g of the form $g = \alpha\bar{z} + \bar{\beta}$ where $g^{-1} = \frac{\bar{z}-\bar{\beta}}{\alpha} = \frac{\bar{z}-\beta}{\alpha}$. Then, $(g^{-1}\circ t^m \circ g)(z) = g^{-1}\circ((t^m\circ g)(z)) = g^{-1}(t^m(g(z))) = g^{-1}((\alpha\bar{z}+\beta)+m) = \frac{((\bar{\alpha}\bar{\bar{z}}+\bar{\beta})+\bar{m})-\beta}{\alpha} = \frac{((\alpha z+\beta)+m)-\beta}{\alpha} = z+\frac{m}{\alpha} \subseteq G$. Therefore, for every $g \in G$, we have shown $g^{-1}\circ T\circ g \subseteq T$, which proves $T \triangleleft G$. □

8. Isometries Congruent mod T

8.1. **General Congruent Form.** If H is a subgroup of G and $x, y \in G$, then x and y are congruent mod H (denoted $x \equiv y(\text{mod}H)$) if $y^{-1}x \in H$

Then when examining frieze groups, we can suppose f and g are congruent mod T. This implies $g^{-1}f = \tau^m$ for some $m \in \mathbb{Z}$. This tells us that $f(z) = g(t^m(z))$. Using this property, we can explore which isometries are congruent mod T.

Proposition 8.1. *If f and g are congruent mod T then either*
$f(z) = \alpha_1 z + \beta_1$ and $g(z) = \alpha_2 z + \beta_2$ where $\alpha_1 = \alpha_2$ or
$f(z) = \alpha_1\bar{z} + \beta_1$ and $g(z) = \alpha_2\bar{z} + \beta_2$ where $\alpha_1 = \alpha_2$

Proof. Case 1: f is of the form $f(z) = \alpha_1 z + \beta_1$ and g is of the form $g(z) = \alpha_2 z + \beta_2$

Since f and g are congruent, $f(z) = g(t^m(z))$. Therefore, $f(z) = \alpha_2(z+m)+\beta_2 = \alpha_2 z + \alpha_2 m + \beta_2$. This tells us that in order for f and g to be congruent, $f(z)$ must equal $\alpha_2 z + \alpha_2 m + \beta_2$. That is, $f(z) = \alpha_1 z + \beta_1 = \alpha_2 z + \alpha_2 m + \beta_2$. This must hold true for all $z \in \mathbb{C}$, so we let $z = 0$ and see $\beta_1 = \alpha_2 m + \beta_2$ which is equivelant to 0 $= \alpha_2 m + \beta_2 - \beta_1$. Returning to the previous form of the equation, we see $\alpha_1 z = \alpha_2 z + (\alpha_2 m + \beta_2 - \beta_1) = \alpha_2 z + 0 = \alpha_2 z$. This tells us that when both f and g use z rather than $\bar{z}$, they must also use the same α in order to possibly be congruent mod T

For example, we let f be a 180^0 rotation so that $\alpha_1 =$ -1. Then g can only be congruent mod T to f if $\alpha_2 =$ -1, so that g is also a 180^0 rotation.

The same holds for $\alpha_1 = \alpha_2 = 1$.

Case 2: f is of the form $f(z) = \alpha_1\bar{z} + \beta_1$ and g is of the form $g(z) = \alpha_2\bar{z} + \beta_2$

Since f and g are congruent, $f(z) = g(t^m(z))$. Therefore, $f(z)$ equals $\alpha_2(\overline{z+m}) + \beta_2$ or $\alpha_2(\overline{z+m+\frac{1}{2}}) + \beta_2$, which is equivalent to $\alpha_2(\bar{z} + m) + \beta_2$ or $\alpha_2(\bar{z}+m+\frac{1}{2}) + \beta_2$. Then, $f(z)$ equals $\alpha_2\bar{z} + \alpha_2 m + \beta_2$ or $\alpha_2\bar{z} + \alpha_2 m + \alpha_2(\frac{1}{2}) + \beta_2$, respectivly. This must hold true for all $z \in \mathbb{R}$, so we let $z = 0$. It follows that $\bar{z} = 0$ and we see β_1 equals $\alpha_2 m + \beta_2$ or $\alpha_2 m + \alpha_2(\frac{1}{2}) + \beta_2$ which is equivalent to saying 0 equals

$\alpha_2 m + \beta_2 - \beta_1$ or $\alpha_2 m + \alpha_2(\frac{1}{2}) + \beta_2 - \beta_1$, respectivly. Returning to the previous form of the equation, we see $\alpha_1\bar{z}$ equals $\alpha_2\bar{z} + (\alpha_2 m + \beta_2 - \beta_1)$ or $\alpha_2\bar{z} + (\alpha_2 m + \alpha_2(\frac{1}{2}) + \beta_2 - \beta_1)$, respectivly, which tells us $\alpha_1\bar{z} = \alpha_2\bar{z}$. This tells us that when both f and g use $\bar{z}$ rather than z, they must also use the same α in order to possibly be congruent mod T.

For example, suppose f is a horzontal reflection so that $\alpha_1 = 1$. Then g can only be congruent mod T to f if $\alpha_2 = 1$, so g is also a horizontal reflection. The same holds for $\alpha_1 = \alpha_2 =$ -1.

Case 3: Either f or g has z, while the other has $\bar{z}$

Without loss of generality, let $f = \alpha_1 z + \beta_1$ and $g = \alpha_2\bar{z} + \beta_2$. Then, $f(z) = g(t^m(z)) = \alpha_2(\bar{z} + \bar{m}) + \beta_2 = \alpha_2(\bar{z} + m) + \beta_2 = \alpha_2\bar{z} + \alpha_2 m + \beta_2$. This must hold true for all $z \in \mathbb{C}$, so we let $z = 0$ which tells us $\bar{z} = 0$ and see $\beta_1 = \alpha_2 m + \beta_2$ which implies $0 = \alpha_2 m + \beta_2 - \beta_1$. Returning to the previous form of the equation, we see $\alpha_1 z = \alpha_2\bar{z} + (\alpha_2 m + \beta_2 - \beta_1) = \alpha_2\bar{z} + 0 = \alpha_2\bar{z}$. This must old true for all $z \in I\mathbb{C}$. However, if we let $z = 1 + i$, it follows that $\bar{z} = 1 - i$. Without loss of generality, let $\alpha_1 = 1$. We then see $\alpha_1 z = 1(1 + i) = 1 + i \neq \alpha_2(1 - i) = \alpha_2\bar{z}$. Therefore, if $f(z) = \alpha_1 z + \beta_1$ and $g(z) = \alpha_2\bar{z} + \beta_2$, f and g can not be congruent mod T. The same holds true if $g(z) = \alpha_2 z + \beta_2$ and $f(z) = \alpha_1\bar{z} + \beta_1$.

We conclude that for all 5 isometries, any two of different types are not congruent mod T □

Proposition 8.2. *If $f \in G$, then $f^2 \in T$*

Proof. Given any $f \in G$ and knowing $T = \langle\tau\rangle \leq G, f$ can be one of five possible cases.

Case 1: f is a translation.

Then, for some $m \in \mathbb{Z}$

$$\begin{aligned} f(z) &= z + m \\ f &= \tau^m \\ f^2 &= \tau^{2m} \in T \end{aligned}$$

Case 2: f is a rotation by 180°.

Then, for some $\beta \in \mathbb{R}$

$$\begin{aligned} f(z) &= -z+\beta \\ f^2(z)=f(f(z)) &= (z-\beta)+\beta \\ &= z \\ Thus, f^2 &= \tau^0 \in T \end{aligned}$$

Case 3: f is a reflection about the x-axis.

Then, for some $m \in \mathbb{Z}$

$$\begin{aligned} f(z) &= \bar{z}+m \\ f^2(z)=f(f(z)) &= \overline{(\bar{z}+m)}+m \\ &= z+\bar{m}+m \\ &= z+2m \\ Thus, f^2 &= \tau^{2m} \in T \end{aligned}$$

Case 4: f is a glide reflection.

Then, for some $m \in \mathbb{Z}$

$$\begin{aligned} f(z) &= \bar{z}+1/2+m \\ f^2(z) &= f(f(z)) = (\overline{\bar{z}+1/2+m})+1/2+m \\ &= z+1+2m \\ Thus, f^2 &= \tau^{2m+1} \in T \end{aligned}$$

Case 5: f is a reflection about the y-axis.

Then, for some $\beta \in \mathbb{R}$

$$\begin{aligned} f(z) &= -\bar{z}+\beta \\ f^2(z)=f(f(z)) &= -\overline{(-\bar{z}+\beta)}+\beta \\ &= z-\bar{\beta}+\beta \\ &= z \\ Thus, f^2 &= \tau \in T \end{aligned}$$

□

9. Quotient Groups

For $H \triangleleft G$, we denote the set of cosets of H as the quotient group G/H equals $(gH \mid g \in G)$ together with an operator given by $gH \bullet fH = gfH$ where $g, f \in G$. It is also easy to check that every quotient group is a group, by going through the requirements for a group.

9.1. **Isometry Correspondance to the Quotient Group.** Recall earlier, when we showed that any two isometries of the same type are congruent mod T, and any two isometries of different types are not. Therefore, each element of G/T corresponds to one type of isometry This tells us that the order of G/T must be less than or equal to five, because there only five different types of isometries.

9.2. **LaGrange's Theorem Applied.** LaGrange's theorem states that for any finite group G, the order of any subgroup H of G must divide the order of G

We have shown that each element of G/T has order one or two. Each of these elements generate cyclic subgroups of G/T, which is finite. Therefore, we can apply LaGrange's theorem, which tells us that the order of G/T must be one, or an even number. From earlier in this section, we saw that the order of G/T is less than or equal to five. Hence, we see that the order of G/T must be either one, two, or four.

10. Isometry Groups

Because of LaGrange's Theorem, we know that the order of G/T must be either one, two, or four. We also know that G/T must contain T. This simply means that each frieze group will have an infinite number of translations. Each order that is not order one will contain any of the other four isometries of G: $\rho(z) = -z + \beta_1 (180^o$ rotation), $v(z) = -\bar{z} + \beta_2$ (verticle reflection), $h(z) = \bar{z}$ (horizontal reflection),or $g(z) = \bar{z} + \frac{1}{2} + m$ (glide reflection).

10.1. **Groups of Order One.** $< T >$

10.2. **Groups of Order Two.** Because T is included in each of the orders of G/T, there are only four possibilites for groups of order two. $< T, vT >< T, hT >< T, \rho T >< T, gT >$

10.2.1. $< T, vT >$.

10.2.2. $< T, hT >$.

10.2.3. $< T, \rho T >$.

10.2.4. $< T, gT >$.

10.3. **Groups of Order Four.** The combinations of order four become very limited for two reasons. The first reason is because T must be included in each goup. The second is because g and h, that is, the glide reflections and horizontal reflections, cannot be included together in the same isometry group. A reflection composed with a glide reflection would result in a translation by $\frac{1}{2}$ or $m + \frac{1}{2}$,for some $m \in \mathbb{Z}$, but that is not included in T, the translations in G. All translations in G are by an integer, so any combination involving a glide refleciton and a horizontal reflection would not work. This only leaves two possible groups of order four.

10.3.1. $< T, vT, \rho T, gT >$. This group consists of vertical reflections, rotations, glide reflections and translations

10.3.2. $< T, vT, \rho T, hT >$. This group consists of vertical and horizontal reflecitons, rotations, and translations.

11. Conclusion

In conclusion, frieze patterns can be represented by isometries in the complex plane. When analyzing these isometries mathematically, we found that there are only seven different possible types of frieze patterns.

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