Realizing Zero-Divisor Graphs

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Acknowledgments

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Outline

What is a Ring?

A **ring** R is a set together with two binary operations $+$ and $\cdot$ (called addition and multiplication) satisfying the following axioms:

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A. R is an Abelian group under $+$
B. For any a, b in R, $a \cdot b$ is in R. *(closure of multiplication)*
C. For any a, b, c in R, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$. *(associativity of multiplication)*
D. For any a, b, c in R, $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$. *(distributive property)*

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Additionally, R is **commutative if for all a, b in R, $a \cdot b = b \cdot a$; and R has **unity** if 1 is in R such that $a \cdot 1 = 1 \cdot a$ for all a in R.

Zero Divisors

Definition

A non-zero element, r, of a commutative ring, R, is called a **zero divisor** if $r \cdot s = 0$ for some non-zero s also in R. We say that r **annihilates** s and vice versa.

Example

Consider the ring $\mathbb{Z}/6\mathbb{Z}$ which has elements $\{0, 1, 2, 3, 4, 5\}$. The zero-divisors are $\{2, 3, 4\}$.

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$$\begin{aligned} 2 \cdot 3 = \quad 6 \quad &\equiv 0 \bmod 6 \\ 3 \cdot 4 = \quad 12 \quad &\equiv 0 \bmod 6 \end{aligned}$$

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An **Anderson-Livingston zero-divisor graph** of a commutative ring, R, with unity is a simple graph (i.e. with no loops or multiple edges) whose set of vertices consists of all non-zero zero divisors, with an edge between a and b if $a \cdot b = 0$. These graphs will be denoted $\Gamma(R)$.

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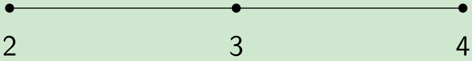

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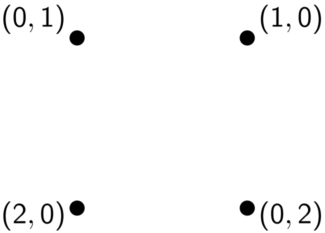

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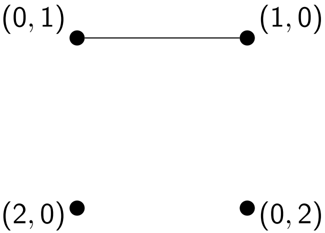

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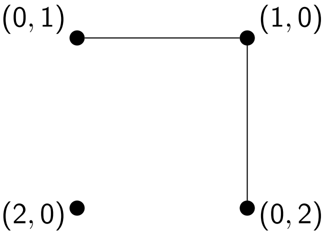

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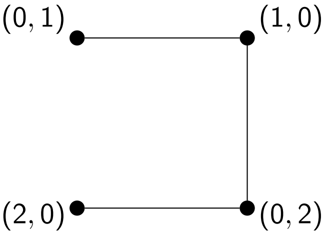

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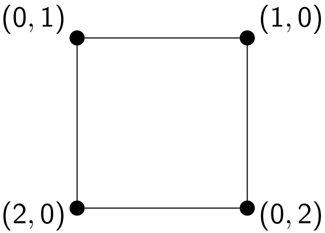

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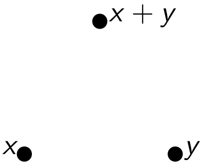

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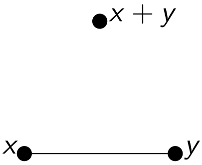

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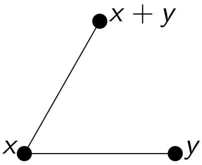

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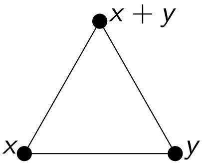

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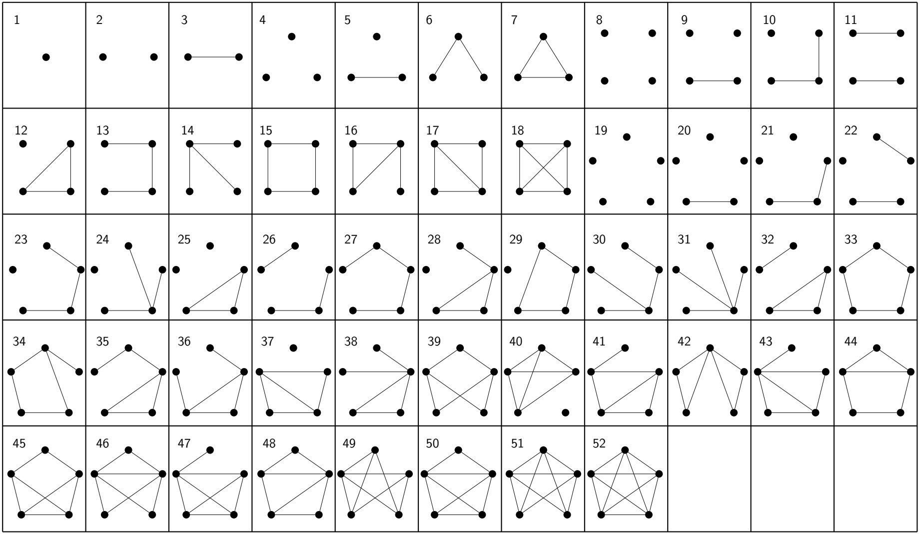
- Draw all the graphs on 1-5 vertices.
- Determine which of these graphs can be realized as the zero-divisor graph of a ring, R.
- Give examples of rings associated with these possible graphs.
- Provide proofs for graphs which cannot be realized as a zero-divisor graph of a ring.

The Graphs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The following graph theory terms will help lessen the load:

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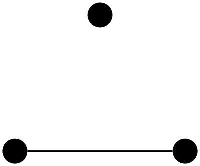
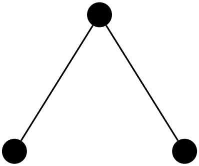


not connected



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The **diameter** of a graph, G, denoted $diam(G)$, is the greatest distance between two vertices (i.e. the maximal number of edges between two vertices).

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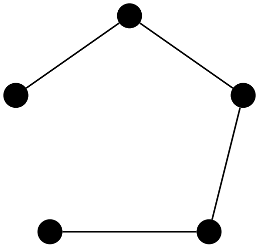
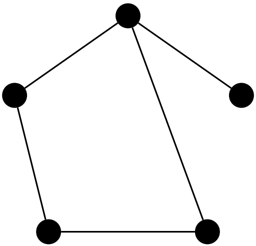


$diam(G) > 3$



$diam(G) \leq 3$

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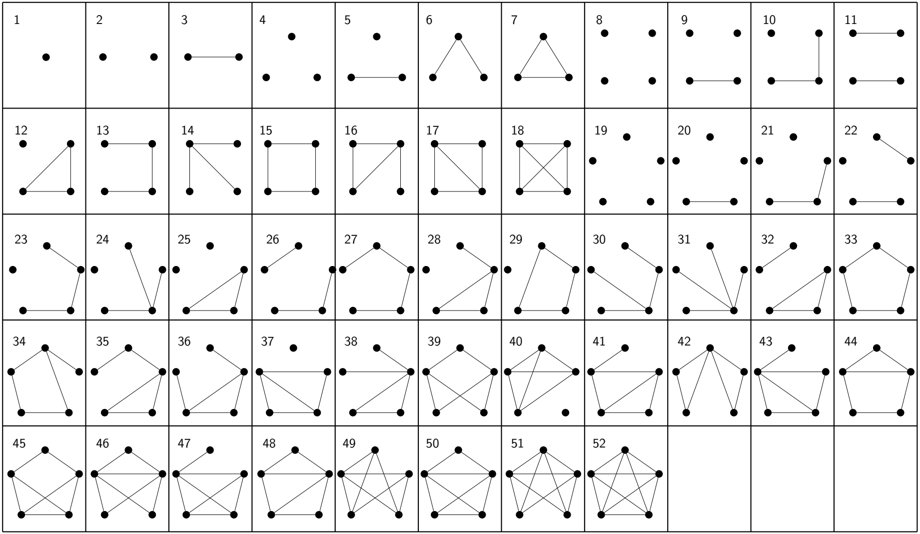
Theorem (Anderson-Livingston)

$\Gamma(R)$ *is always connected.*

Theorem (Anderson-Livingston)

$diam(\Gamma(R)) \leq 3$

New Set of Graphs



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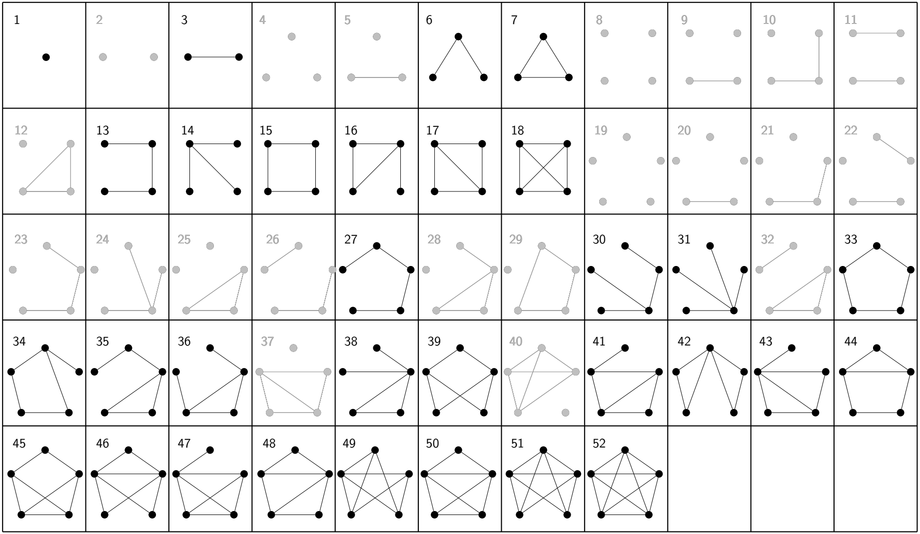

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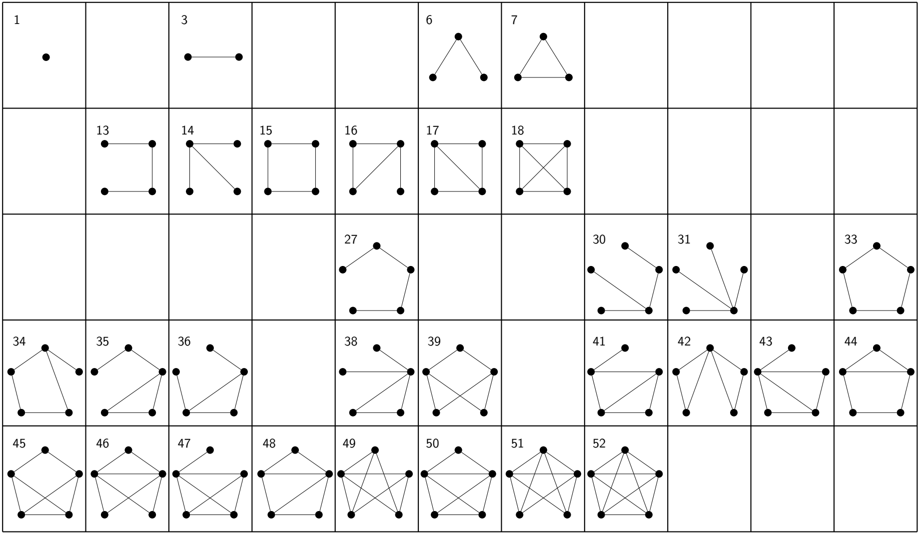

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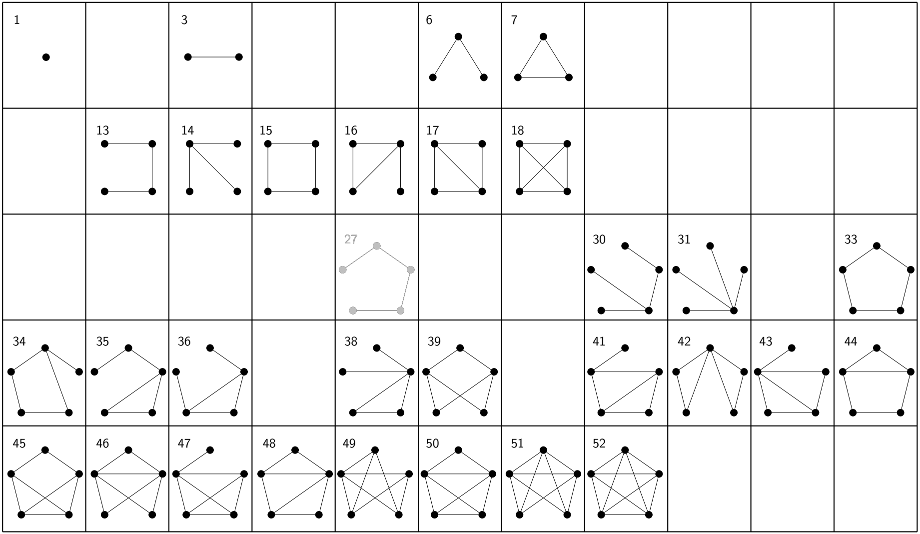

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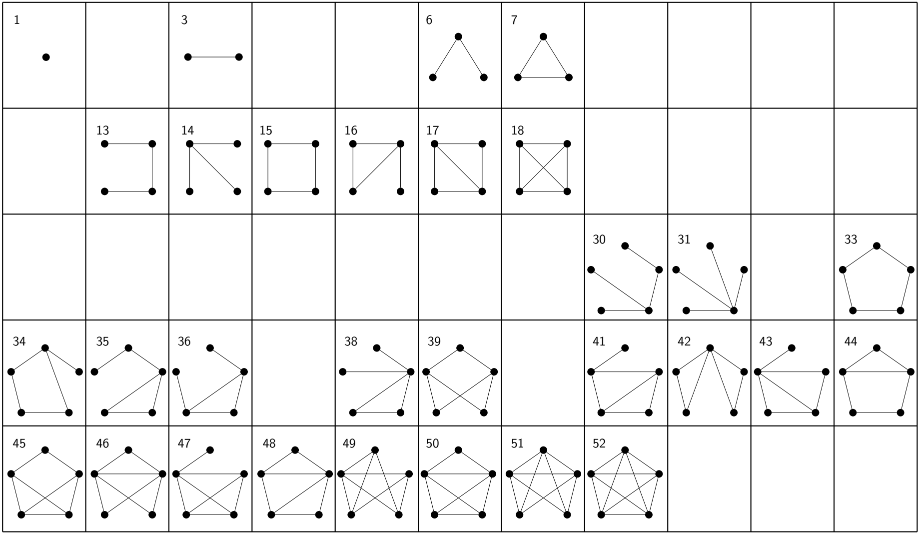

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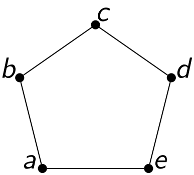
  - Etc.

Non-Existence Proof



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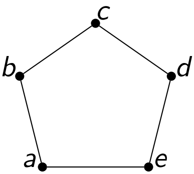

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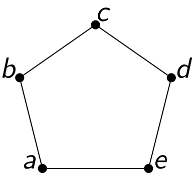

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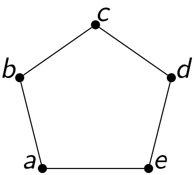

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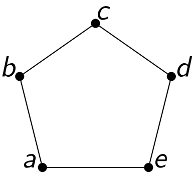

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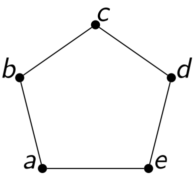

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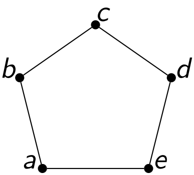

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- No element is annihilated by b, c, and e. Therefore, this cannot be $\Gamma(R)$ for any commutative R. $\square$

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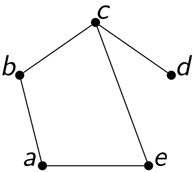

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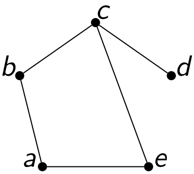

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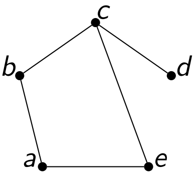

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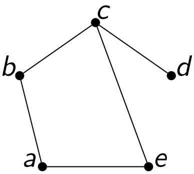

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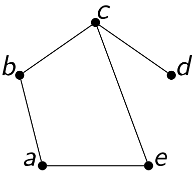

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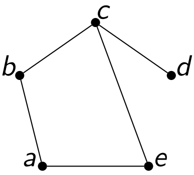

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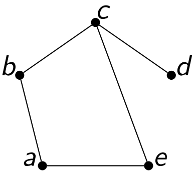

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This graph cannot be realized as a zero-divisor graph of a ring.

Proof.

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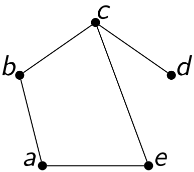

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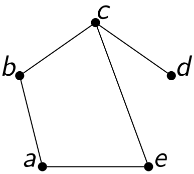

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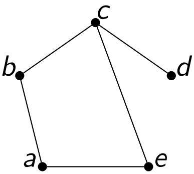

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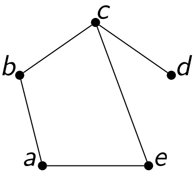

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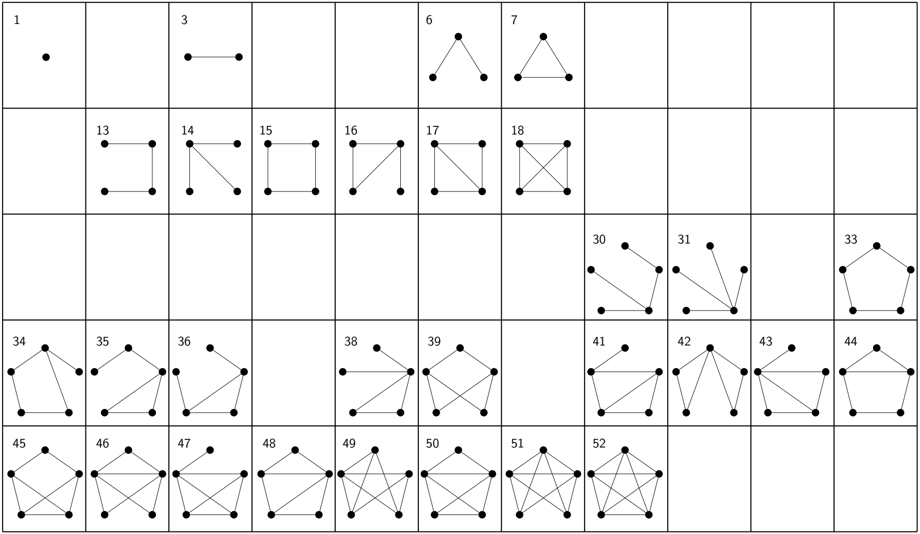
- Therefore, this graph is not $\Gamma(R)$ for any commutative R. $\square$

The Remaining Graphs



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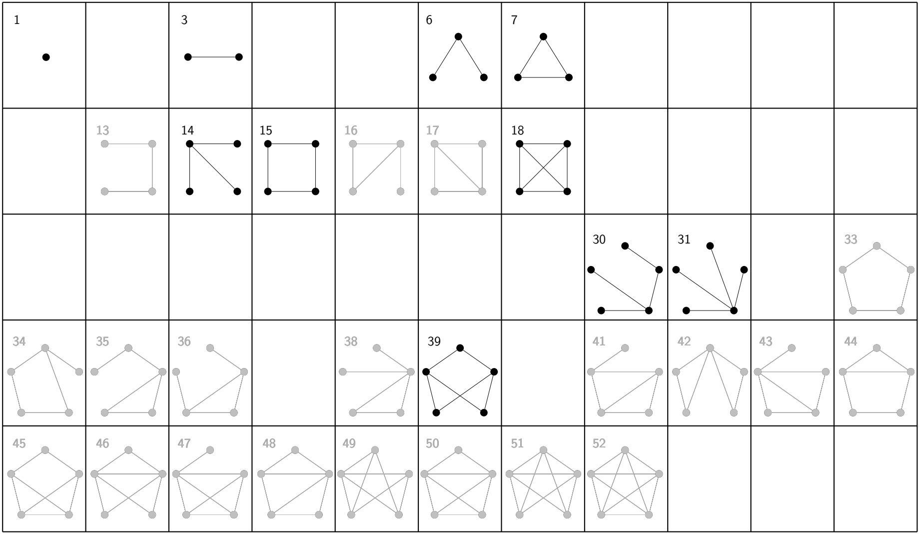

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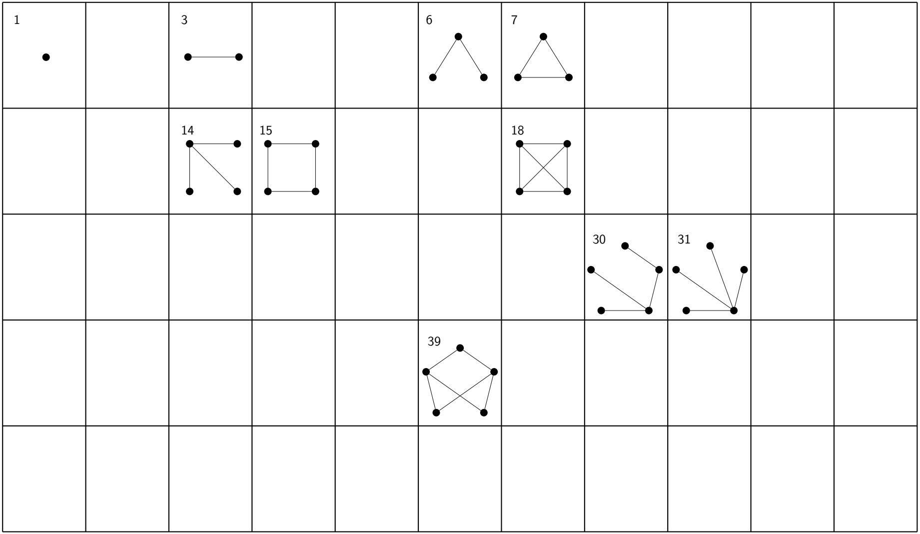
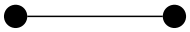
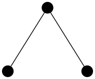
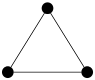
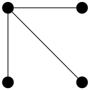
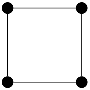
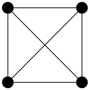
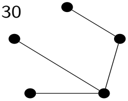
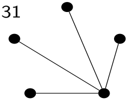
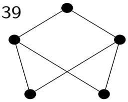


Graphs Realized as AL Graphs

1 	$\mathbb{Z}_4$, $\frac{\mathbb{Z}_2[x]}{<x^2>}$
3 	$\mathbb{Z}_9$, $\mathbb{Z}_2 \times \mathbb{Z}_2$ $\frac{\mathbb{Z}_3[x]}{<x^2>}$
6 	$\mathbb{Z}_6$, $\mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{<x^3>}$
7 	$\frac{\mathbb{Z}_2[x,y]}{<x^2,xy,y^2>}$
14 	$\mathbb{Z}_2 \times \mathbb{F}_4$ $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{<x^2>}$

15 	$\mathbb{Z}_3 \times \mathbb{Z}_3$
18 	$\mathbb{Z}_{25}$, $\frac{\mathbb{Z}_5[x]}{<x^2>}$
30 	$\mathbb{Z}_2 \times \mathbb{Z}_4$
31 	$\mathbb{Z}_{10}$
39 	$\mathbb{Z}_3 \times \mathbb{F}_4$

Mulay Graphs

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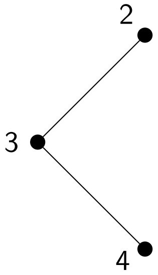

$\Gamma(\mathbb{Z}_6)$

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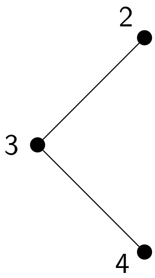

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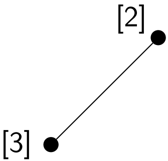

$\Gamma_E(\mathbb{Z}_6)$

Larger Graphs

Mathematicians have discovered the possible [AL] zero-divisor graphs of up to 14 vertices. Past that, there are a significantly larger number of graphs to consider.

Work can be done to find more ways of being able to eliminate more "types" of graphs.