)15)25 )36)46 )57)67 )78)88 )98)109 )119)1210 )1310)1411

# ZERO DIVISOR GRAPHS

CARLOS LOPEZ, ALONZA TERRY, AND ALAINA WICKBOLDT

Abstract. We define and study the zero divisor graphs, in particular the Anderson and Livingston zero divisor graphs. We will discuss how to create them and what methods we used to find patterns in these graphs. We also discuss the behavior of the ring as a whole and of it's parts.

## 1. Introduction

1.1. **Creating a Zero Divisor Graph.** During this project we studied the properties of zero divisor graphs. First we will define some Ring Theory and then some Graph theory. During this project we focused mainly on the Anderson and Livinston graphs. Then we will discuss the methods we used to come to our conclusions. Then we will discuss our findings and provide an analysis for the different types of behavior of these graphs. We only looked at graphs for the integers modulo 100.

## 2. Background

### 2.1. Ring Theory Definitions.

(1) **ring** $R$ is a set together with two binary operations $+$ and $\cdot$ (called addition and multiplication) satisfying the following axioms:
   A. $R$ is an Abelian group under $+$; i.e.,
      i. For any $a, b$ in $R$, $a+b$ is in $R$. *(closure of addition)*
      ii. For any $a, b, c$ in $R$, $a + (b + c) = (a + b) + c$. *(associativity of addition)*
      iii. There exists 0 in $R$ such that $a + 0 = 0 + a = a$ for all $a$ in $R$. *(additive identity)*
      iv. For any $a$ in $R$, there exists an element $b$ in $R$ such that $a + b = b + a = 0$. *(additive inverses)*
      v. For all $a, b$ in $R$, $a + b = b + a$. *(abelian condition)*

---

1991 *Mathematics Subject Classification.* 05B35.

B. For any $a, b$ in $R$, $ab$ is in $R$. *(closure of multiplication)*
C. For any $a, b, c$ in $R$, $a(bc) = (ab)c$. *(associativity of multiplication)*
D. For any $a, b, c$ in $R$, $a(b+c) = ab+ac$ and $(a+b)c = ac+bc$. *(distributive property)*

(2) $R$ has **unity** if 1 is in $R$ such that $a \cdot 1 = 1 \cdot a$ for all $a$ in $R$.
(3) A ring is **commutative** if multiplication is commutative.
(4) A **unit** is a non-zero element u in a commutative ring with 1, such that there is another non-zero element v of the ring satisfying u$*v = 1$.
(5) A **zero divisor** of a commutative ring is a non-zero element r such that $rs = i$ for some other non-zero element s of the ring. If the ring R is commutative, then $rs = 0 \Leftrightarrow sr = 0$.
(6) An **integral domain** is a commutative ring within 1 such that $R$ has no zero divisors.
(7) A **field** is an integral domain in which every non-zero element is a unit.

2.2. **Graph Theory Definitions.**

(1) A **graph** consists of a set of vertices, a set of edges, and an incident relation describing, which vertices are adjacent (i.e., joined by an edge) to which. $G = (V, E)$.
(2) A **path of length n** between two vertices v and w is a finite sequence of vertices $u_0, u_1, ..., u_n$ such that $v = u_0$, $w = u_n$, and $u_{i-1}$ and $u_i$ are adjacent for all $1 \leq i \leq n$.
(3) A graph is **connected** if there is a path between every pair of vertices of the graph.
(4) The **diameter** of a connected graph is the greatest distance between any two vertices, where, by "greatest distance," we mean how many edges we "must" travel.
(5) The **degree** of a vertex is the number of edges issuing from it.
(6) A **vertex** is called a end (leaf, pendent) if the degree is 1.
(7) A graph is **complete**, if every vertex in the graph is adjacent to each other vertex in the graph.
(8) A **cycle** in a graph is a path V$_1, V_2, ..., V_k$[k $\geq 3$] together with the edge $V_kV_1$.
(9) If a graph is a cycle, then the graph is called a **cycle graph**.
(10) A graph which does not contain a cycle is called **acyclic**.

2.3. **Definitions of Zero Divisor Graphs.**

(1) (Beck) The **zero divisor graph** of a commutative ring $R$ with 1 is a simple graph whose set of vertices consists of all elements

of the ring, with an edge defined between $a$ and $b$ if and only if $ab = 0$.

(2) (Anderson and Livingston) $\Gamma(R)$ The idea of Becks simple graphs are further expanded where zero divisors are represented. Therefore, any element of the ring which is not a zero divisor is not shown. the diameter is three.

(3) (Mulay) $\Gamma_E$ The **zero divisor graph is determined by equivalence classes of commutative ring** is a simple graph whose set of vertices consisting of all equivalence classes of zero divisors (where $a\tilde{b} \Leftrightarrow ann(a) \subset ann(b)$) with an edge defined between $[a]\&[b] \Leftrightarrow a\tilde{=}0$ for any representation of the classes.

(4) A graph is **complete bipartite** is a graph whose vertex set can be partitioned into 2 disjoint subset $(u_i)$ and $(v_j)$ such each ui is adjacent to every $vj$, but no two ui's are adjacent but no two vj's are adjacent.

## 3. Our Methods

We were presented with the task to observe the behavior of various Anderson and Livingston Zero Divisor Graphs. Our goal was to study the patterns of these graphs. At first, the way we approached this task was by drawing as many graphs as we could. After many tedious drawings we realized that once you got into larger numbers, the graphs became exceedingly complex. Another task was the shape the graphs in order to give a more clear appearance. Although some large numbers were exceedingly difficult to draw by hand, some were fairly easy. For example, the number 33. This graph and others like $\Gamma(\mathbb{Z}_{22})$ and $\Gamma(\mathbb{Z}_{14})$ lead us to our first conclusion; drawing the graphs of two primes multiplied together is relatively easy. Is consists of a complete bipartite graph where one side has multiples of the first prime and the other had multiples of the second prime.

This conclusion lead us to believe that the graphs would not be more complex simply because they were larger, but because of the complexity of their prime factorizations.

As we began shaping more and more of the graphs the evidence confirmed our conjecture. So we began the task of grouping numbers by their prime factorizations. This took a while to complete but finally we came up with 15 different groupings which are explained in detail below.

After clarifying these categories we then observed that drawing a Zero Divisor Graph for any number is not as complex as we thought. Each category can be broken down into a series of steps that are in the following section.

The we discovered that Brendan Kelly and Elizabeth Wilson had developed some mathematica code that could be used to find any zero divisor graph desired. This sped up the process of finding patterns and also confirmed our conjectures.

Our mentor Dave Chapman led us to a program called GAP4. We used this program to more efficiently process our observations of the groupings based on prime factorizations. Using our information from drawings and GAP4, we began to rearrange the mathematica code w ewere given in order to take our loops. After this accomplishment, our mentor and Professor Sandra Spiroff began to inquire if we might be able to write a program to produce a graph that some would claim is a more proficient graph to use, the Mulay Graph. Using our previous program, we began manipulating the coding and came across an even more glorious success. We were able to produce a Mulay Graph for any inputted number $n$.

## 4. Patterns and Findings

We found some interesting patterns in these Anderson and Livinston graphs.The notation used for Anderson and Livingston Graphs is $\Gamma(\mathbb{Z}_n)$. The best way to break these down into categories is by looking at their prime factorization. The more complex the prime factorization, the more complex the graph will be. Let $p$ and $q$ and $r$ represent three distinct primes for all of the following cases.

### 4.1. **One Prime.**

When drawing the Anderson and Livingston graph this is trivial because we eliminate the zero divisors 0 and 1. These graphs have no vertices or edges.

All numbers in this category $< 100$ are $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37,$

41,43, 47,53,59,61,67,71,73,79,83,89, and 97.

### 4.2. **Two Primes.**

Let this be represented by $pq$. There are two cases for this, $p$ and $q$ can be distinct or they can be the same.

#### 4.2.1. *The distinct case.*

If $p$ and $q$ are distinct then the graph will be a complete bipartite graph. On one side of the graph we can list all the factors of the first prime number and on the other side we can list all the factors of the second prime. Then we construct a complete bipartite graph.

Example. $\Gamma(\mathbb{Z}_{33})$

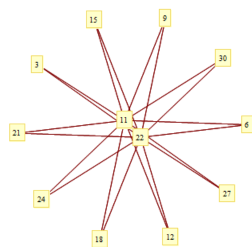


FIGURE 1. $\Gamma(\mathbb{Z}_{33})$

All numbers in this category $< 100$ are $6, 10, 14, 15, 21, 22, 26, 33, 34, 35, 38,$ 39,46,51,55,57,58,62,65,69,74,77,82,85,86,87,91,93,94,95.

4.2.2. *The non-distinct case.* Take the case $p^2$. This will be a complete graph with $p-1$ vertices. Every vertex will be a multiple of $p < p^2$.

Example. $\Gamma(\mathbb{Z}_{25})$

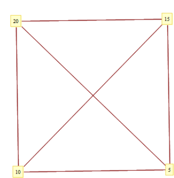

FIGURE 2. $\Gamma(\mathbb{Z}_{25})$

All numbers in this category $< 100$ are $9, 25$, and $49$.

4.3. **Three Primes.** This can be represented three different ways. $p^3$, $p^2q$, and $pqr$.

4.3.1. *The $p^3$ case.* This graph will be a complete bipartite graph with one side having one set of vertices that are multiples of $p^2$. The other set of vertices being all remaining multiples of $p$.

Example. $\Gamma(\mathbb{Z}_{27})$

All numbers in this category $< 100$ are 8 and 27.

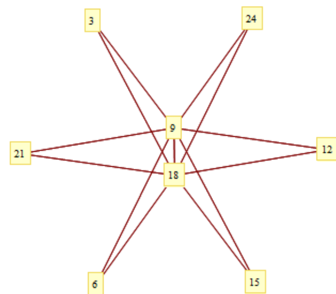


FIGURE 3. $\Gamma(\mathbb{Z}_{27})$

4.3.2. *The $p^2q$ case.* The center of this graph will be $(p^2q)/2$. The vertices connected to $(p^2q)/2$ will be multiples of $pq$. These multiples will form a complete bipartite graph with the multiples of p that are remaining. Connected to $(p^2q)/2$ will be all remaining multiples of $q$. The tail vertices are $qr$ where $r$ is relatively prime to $p$. The head vertices are $rp$ where $r$ is relatively prime to $q$. The center node is $3^2$, or $p^2$. The gill vertices are multiples of $pq$.

Example. $\Gamma(\mathbb{Z}_{18})$

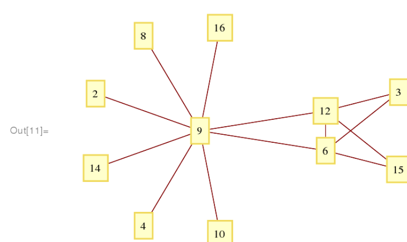


FIGURE 4. $\Gamma(\mathbb{Z}_{18})$

Note that all of the vertices on the right are combinations of the primes $p$ and $q$.

All numbers in this category $< 100$ are $12, 18, 20, 28, 44, 45, 52, 76,$

$50,63,68,75,92,98 and 99$.

4.3.3. *The $pqr$ case.* The first number like this is 30. On the right side of the graph are the multiples of 2 that do not contain a factor of 3. At the center is half of $pqr$. Each grouping of terms is a group of factors that have to do with the multples of $p$, not associated with $q$ or $r$, then the multiples of $r$ not associated with $p$ or $q$, and then the multiples of $pq$.

Example. $\Gamma(\mathbb{Z}_{30})$

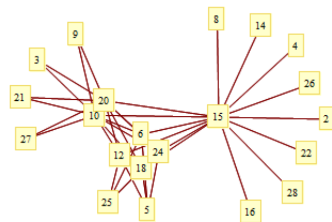


FIGURE 5. $\Gamma(\mathbb{Z}_{30})$

All numbers in this category $< 100$ are $30, 42, 66, 70$, and 78.

4.4. **Four Primes.** This can be represented four different ways. $p^4$, $p^3q$ $p^2qr$, and $p^2q^2$.

4.4.1. *The $p^4$ case.* The center of this graph will be multiples of $p^3$.One one side of these vertices will be a complete graph with vertices of all "'unused"' multiples of $p^3$. On the other side side of these vertices will be a complete bipartite graph with all remaining multiples of 3.

Example. $\Gamma(\mathbb{Z}_{81})$

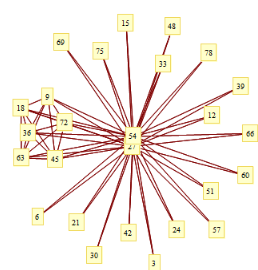

FIGURE 6. $\Gamma(\mathbb{Z}_{81})$

All numbers in this category $< 100$ are 16 and 81.

4.4.2. *The $p^3q$ case.* This graph is very complex. We noticed that the higher the power that the prime is raised to, the more complex the graph will be if it is combined with two different primes in it's prime factorization. Again each category can be broken down into multiples of $p, q, pq, p^2qp^3$,and $pq$

Example. $\Gamma(\mathbb{Z}_{24})$

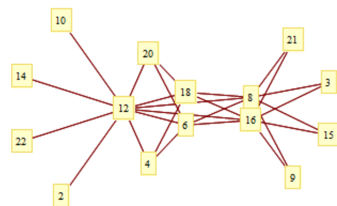


FIGURE 7. $\Gamma(\mathbb{Z}_{24})$

All numbers in this category $< 100$ are $24, 40, 54$, and $56$.

4.4.3. *The $p^2qr$ case.* Example. $\Gamma(\mathbb{Z}_{60})$

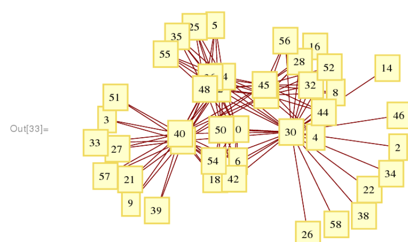


FIGURE 8. $\Gamma(\mathbb{Z}_{60})$

4.4.4. *The $p^2q^2$ case.* Example. $\Gamma(\mathbb{Z}_{36})$

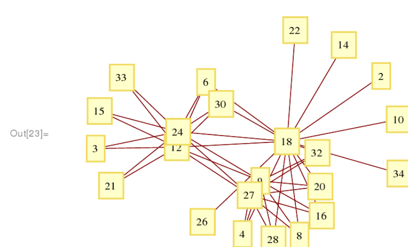


FIGURE 9. $\Gamma(\mathbb{Z}_{36})$

All numbers in this category $< 100$ are 36 and 100

4.5. **Five Primes.** There are three categories to look at where the number is less than 100; $p^5$,$p^4q$ and $p^3q^2$. Another pattern we noticed is that when there is one prime raised to a high power, the graph is more simple than if there are two distinct primes raised to an exponent. An example below is $\Gamma(\mathbb{Z}_{32})$. This graph is less complex than $\Gamma(\mathbb{Z}_{24})$ because 32 is one prime (2) raised to the $5th$ power, while 24 is two distinct primes ($2^3$ and 3) combined.

4.5.1. *The $p^5$ case.* This is simply a two sided graph with the center being $p^5/2$ and on one side a complete graph with multples of $p^2, p^3, ...$, then on the other side will be all remaining multples of $p$.

Example. $\Gamma(\mathbb{Z}_{32})$

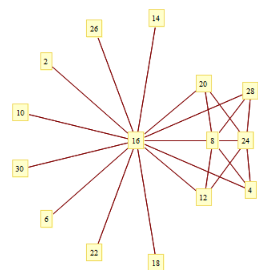

FIGURE 10. $\Gamma(\mathbb{Z}_{32})$

All numbers in this category $< 100$ are 32 and 81.

4.5.2. *The $p^4q$ case.* These graphs are some of the more compolex graphs with $n < 100$ because we are getting into comlex combinations of prime numbers. Example. $\Gamma(\mathbb{Z}_{48})$

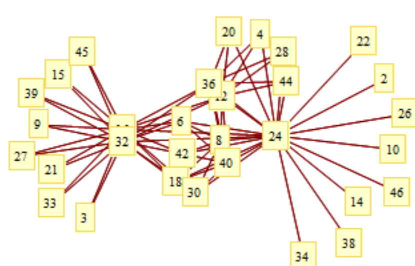


FIGURE 11. $\Gamma(\mathbb{Z}_{48})$

All numbers in this category $< 100$ are $48, 64, 80$.

4.5.3. *The $p^3q^2$ case.* There is only one graph like this.

Example. $\Gamma(\mathbb{Z}_{72})$

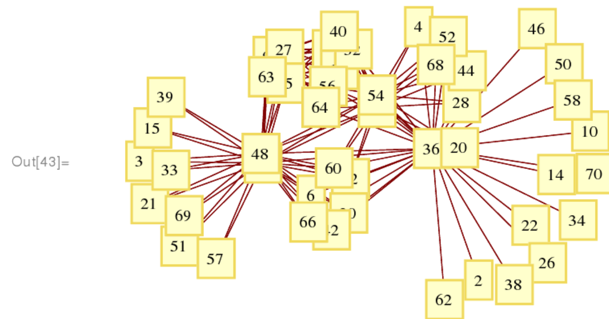


FIGURE 12. $\Gamma(\mathbb{Z}_{72})$

4.6. **Six Primes.** There are only two categories to look at here where n is $< 100$

4.6.1. *The $p^6$ case.* The only number that falls into this category $< 100$ is 64.

Example. $\Gamma(\mathbb{Z}_{64})$

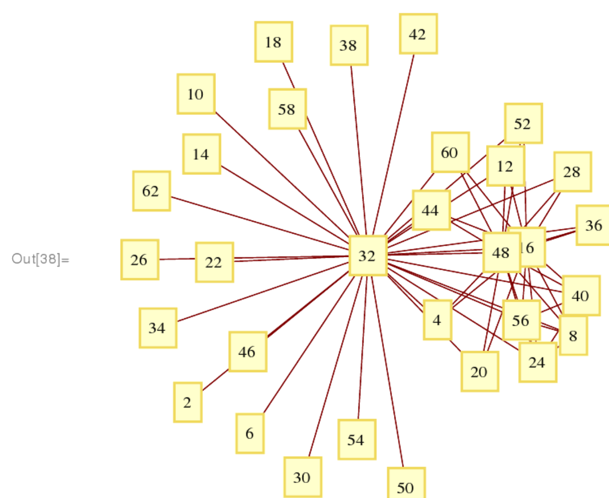


FIGURE 13. $\Gamma(\mathbb{Z}_{64})$

4.6.2. *The $p^5q$ case.* The only number that falls into this category $<$ 100 is 96. This is also the most complex graph that we will deal with because it is the highest combination of primes.

Example. $\Gamma(\mathbb{Z}_{96})$

Out[60]=

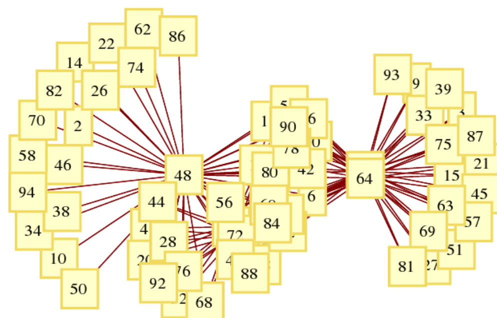


Figure 14. $\Gamma(\mathbb{Z}_{96})$

## 5. Conclusion

We found that our conjecture about the complexity of the graph being directly related tot he complexity of the prime factorization of a number is correct. The fewer primes that a number can be broken down into the less complex the graph will be. Every graph can be construced by hand, or by mathematica programming, but there is a simple procedure for constructing any graph $< 100$ as described in detail above.

## References

[1] D. Anderson, P. Livingston, The zero-divisor graph of a commutative ring, J. ALgebra, 217 (1999)
[2] S.B. Mulay.
[3] Dr. Spiroff, Sandra. University of Mississippi.
[4] Dr. Chapman, Dave. Louisiana State University.
[5] Kelly, Brendan. The College of New Jersey
[6] Wilson, Elizabeth. The College of William and Mary

SMILE Program, Mississippi State University, Baton Rouge, Louisiana

SMILE Program, Xavier University, New Orleans, Louisiana

SMILE Program, Louisiana State University, Baton Rouge, Louisiana