Stability Analysis of an SIR Epidemic Model

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Introduction

- The SIR epidemic model is a dynamical system, as it can vary with respect to time for up to all three variables
- When conducting a stability analysis, we ask the following questions:
- Are there any constant solutions?
- If so, do solutions near the constant move toward or away from the constant solution?
- What happens to solutions as t approaches infinity?
- Do any solutions oscillate?

Introduction

- If there is a constant solution, the phase portarait of the dynamical system will either show the solutions having vectors tending either toward or away from the equilibrium value.
- If the values of the system near the equilibrium value all tend toward the point, the equilibrium point is considered stable, or an attractor.
- If the values of the system near the equilibrium value all tend away from the value, then the point is considered unstable, or a repelling point.
- In some cases, some values tend toward the equilibrium point, and some tend away from it. This is called a saddle point. It is unstable.

Introduction

- Yet other cases show an equilibrium point at the origin, but all trajectories near the equilibrium point stay a small distance away. This is a stable equilibrium point, but it is not globally asymptotically stable.
- The case where both eigenvalues are real, negative, and distinct produces a phase portrait that shows all trajectories tending toward the equilibrium point as $t \to \infty$, the value of $x(t)$ gets small, so it is a globally stable equilibrium point.
- When both eigenvalues are real, negative, and equal, the phase portrait shows a globally stable equilibrium point if the two eigenvectors are linearly independent.

Introduction

- All individuals in a given population are in the S group by birth, but are gradually "removed" by either being infected and quaranitned or being quarantined prior to infection.
- The birth rate is taken into account with the S' group; a constant rate of individuals being added to the susceptible population is assumed.
- The death rate is taken into account with the R' group; the per-capita death-rate is simply multiplied by the number of individuals considered "removed" (it is a function of how many individuals are removed, not constant like the birth rate).

Introduction

- The model takes into account the fact that not all contacts between the susceptible group and the infectious group are "adequate contact" or sufficient to cause infection.
- The "basic reproduction ratio" $\frac{b}{g}$ gives the researcher an idea of how the epidemic will progress:
- If it is greater than 1, the number of individuals infected is greater than the number of individuals recovering, so the disease will continue to spread.
- If it is less than (or equal to) 1, the number of individuals recovering is greater than the number of individuals infected, so the disease will spread more slowly.
- This is a very important parameter in epidemiology.

Introduction

- N is a constant; it is the total number of the population. The derivative of N is zero. This is very important for solving the system of differential equations $0 = S' + I' + R'$.
- The initial value of t is zero, but for the stability analysis we assume t to be the open interval in the neighborhood of (0,0).
- GLOBAL ASYMPTOTIC STABILITY: Equlibrium values for the system approach the same value from all sides when looking at the phase portrait of the dynamical system.
- Phase portrait: a "sketch" of the trajectories of a dynamical system, showing the direction of the motion of the point(x,y) as t increases. The direction is indicated by the velocity vector $\frac{\frac{dx}{dt}}{\frac{dy}{dt}}$.

Introduction

- Why do we use this model?
- Epidemiologists use this model because it allows more flexibility than the logistic model, which allows for only differing birth and death rates.
- It allows the ability to form conclusions regarding the likelihood of a contact resulting in an infection (which tells researchers how quickly people move from the S group to the I group).
- Researchers can also observe the rate where people move from the I group to the R group, showing how quickly individuals move from "infectious" to "removed".
- Once an individual moves into the R group, he or she is incapable of being reinfected or infecting others.

Introduction

An example of this scenario is smallpox:

- every person born (prior to being innoculated, which is not considered here) is equally prone to infection, so included in the S group.
- Once an individual is in the I group, or infectious, they are capable of spreading the disease to others based on the basic reproduction ratio.
- They remain in the I group until they either recover (g) or die (m), both of which occur proportionally to the number of people in the I group.
- Those that are quarantined either before or after infection, and are thus not able to be part of the I class by contributing to the infection of others, are considered removed, or are in the R group.

Project Goal

To properly understand the behavior of the SIR model we must first perform a complete stability analysis of the model. This task includes:

- finding the limit system
- finding upper and lower bounds of the system
- finding an equilibrium point
- applying appropriate theorems to determine local or global stability

Project Goal

- We must also determine whether the equilibrium point is stable or asymptotically stable.
- For our model, we found that the system is **globally asymptotically stable**.

Background Information

Consider the model

$$S'(t) = \Lambda - \beta S \frac{1}{N} - \mu S$$

$$I'(t) = \beta S \frac{1}{N} - (\mu + \gamma) I$$

$$R'(t) = \gamma I - \mu R,$$

where

$$N(t) = S(t) + I(t) + R(t).$$

To perform the stability analysis of our system, we will employ an autonomous system instead of the nonautonomous system we were given.

Definition

Consider the following systems:

$$\frac{d(X)}{dt} = F(t, X) \tag{1}$$

$$\frac{d(Y)}{dt} = G(X) \tag{2}$$

Equation 1 is called **asymptotically autonomous** with the limit system, Equation 2, if $F(t, X) \to G(X)$ as $t \to \infty$ for X in $\mathbb{R}^n$.

Limit System

- Utilizing the given equation $N(t) = S(t) + I(t) + R(t)$, we take the derivative of each side and simplify.
- This yields the solution $N'(t) = \Lambda - \mu N(t)$.
- After differentiation the solution is $N = \frac{\Lambda}{\mu} + \frac{c}{e^{\mu t}}$.
- $lim_{t\to\infty}(\frac{\Lambda}{\mu} + \frac{c}{e^{\mu t}}) = \frac{\Lambda}{\mu}$.
- Therefore, $N = \frac{\Lambda}{\mu}$ for large values of t.
- Thus, we have a limit system, and we can treat N as a constant, $\frac{\Lambda}{\mu}$.

Lower Bound

- $N(t)$ is the population at any given time.
- This must always be positive because a negative population does not make sense. Therefore, $N > 0$.
- This implies that $S(t) > 0$, $I(t) > 0$, and $R(t) \geq 0$.
- Thus, the lower bound for S, I, and R is 0.

Upper Bound

- To find the upper bound of the system, we will use the limit system found earlier.
- As discussed earlier N can now be treated as a constant, $N = \frac{\Lambda}{\mu}$, for large values of t.
- However, for other values of t, $N = \frac{\Lambda}{\mu} + \frac{c}{e^{\mu t}}$.
- S, I, and R are separately $\leq N$.
- Therefore, S, I, and $R \leq \frac{\Lambda}{\mu} + \frac{c}{e^{\mu t}}$.

Confirmation

- We have found the limit system for the model.
- We have found the lower bound (S, I, and R > 0) and the upper bound (S, I, and $R \leq \frac{\Lambda}{\mu} + \frac{c}{e^{\mu t}}$).
- We can now apply the following Theorem :

Theorem

$$\frac{d(X)}{dt} = F(t, X) \tag{3}$$

$$\frac{d(Y)}{dt} = G(X) \tag{4}$$

where Equation 4 is the limit system of Equation 3.

Theorem

If solutions of Equation 3 are bounded and the equilibrium X of Equation 4 is globally asymptotically stable, then any solution X(t) of Equation 3 satisfies $X(t) \to X$ as $t \to \infty$.

Therefore, we may now consider only the limit system.

Equilibrium Point

We now need to find the equilibrium points for these equations.

Definition

Given an equation $\frac{dx}{dt} = f(x)$, a point x^* is an **equilibrium point** if $f(x^*) = 0$.

Equilibrium Point

We can get the equilibrium points by setting the equations in the model equal to zero and solving the system for S, I, and R. By doing so, we obtain:

$$S = \frac{\Lambda^2}{\mu(\beta + \Lambda)}$$

$$I = \frac{\beta\Lambda}{(\beta + \Lambda) + (\mu + \gamma)}$$

$$R = \frac{\gamma\beta\Lambda}{\mu(\beta + \Lambda)(\mu + \gamma)}.$$

Equilibrium Point

- We know that this is the only equilibrium point because the equation for S is linear.
- Therefore, the equilibrium point is:

$$\left(\frac{\Lambda^2}{\mu(\beta+\Lambda)}, \frac{\beta\Lambda}{(\beta+\Lambda)+(\mu+\gamma)}, \frac{\gamma\beta\Lambda}{\mu(\beta+\Lambda)(\mu+\gamma)}\right).$$

Reducing System to Two Equations

- We rewrite $N = S + I + R$ into $R = N - S - I$.
- Substituting this into the system yields:

$$S' = \Lambda - \frac{\beta S}{N} - \mu S,$$

$$I' = \frac{\beta S}{N} - (\mu + \gamma) I,$$

$$R' = I(\gamma + \mu) - \mu(N - S).$$

- R' may be disregarded.

Now that we have simplified our system down to two autonomous equations,

$$S'(t) = \Lambda - \beta S \frac{1}{N} - \mu S$$

and

$$I'(t) = \beta S \frac{1}{N} - (\mu + \gamma) I,$$

we can analyze the stability using the proper Lyapunov function.

Lyapunov Function

Definition

A function $V(x, y)$ is said to be **positive definite** on a region D containing the origin if for all $(x, y) \neq (0, 0)$, $V(x, y) > 0$.
$V(x, y)$ is said to be **negative definite** on a region D containing the origin if for all $(x, y) \neq (0, 0)$, $V(x, y) < 0$.

Definition

A function $V(x, y)$ is said to be a **Lyapunov Function** on an open region D if the function is continuous, positive definite, and has continuous first-order partial derivatives on D.

Theorem

Theorem

If there exists a Lyapunov function $V(x, y)$, dependent on a system $\frac{dx}{dt} = f(x, y)$ and $\frac{dy}{dt} = g(x, y)$ with equilibrim point $(x, y) = (0, 0)$, and $\frac{dV}{dt}$ is negative definite on an open region D containing the origin, then the zero solution of the system is asymptotically stable.

When D encompasses all possible values of (x, y) and follows all of the specified criteria above, the projected stability of the system is said to be global.

V

A possible Lyaponov function that is very common is

$$V = x^2 + y^2.$$

- We are interested in showing not only local, but global asymptotic stability.
- This means that our chosen V will have to satisfy the criteria of a Lyapunov function over the entire region $D = (0, \infty)$x$(0, \infty)$ for which S x I is defined.
- This criteria includes

  i) $V(0,0) = (0,0)$

  ii) $V(x,y) > 0 \; \forall \; (x,y) \neq (0,0)$ on D (positive definite).

Change of Variables

However, this Lyapunov function only works for those systems with an equilibrium point set at the origin, and as can be seen looking at our specified region $D = (0, \infty)\text{x}(0, \infty)$, the origin is not included. This forces us to use a change of variables in order to utilize the function $V = x^2 + y^2$. We take

$$x = S - \frac{\Lambda^2}{\mu(\beta + \Lambda)},$$

$$y = I - \frac{\beta\Lambda}{(\beta + \Lambda)(\mu + \gamma)}.$$

So

$$V = (S - \frac{\Lambda^2}{\mu(\beta + \Lambda)})^2 + (I - \frac{\beta\Lambda}{(\beta + \Lambda)(\mu + \gamma)})^2.$$

Our New D

When redefining our D function for (x,y), it is shifted to the left, leaving us with $D_s = (-\frac{\Lambda^2}{\mu(\beta+\Lambda)}, \infty) \times (-\frac{\beta\Lambda}{(\beta+\Lambda)(\mu+\gamma)}, \infty)$, an open region including the origin.

We now check the derivative of our V:

$$\frac{dV}{dt} = \frac{\partial V}{\partial S}\frac{dS}{dt} + \frac{\partial V}{\partial I}\frac{dI}{dt}$$

$$= -2[(\frac{\mu(\beta+\Lambda)}{\Lambda^2})(S-\frac{\Lambda^2}{\mu(\beta+\Lambda)})^2+(\frac{(\beta+\Lambda)(\mu+\gamma)}{\beta\Lambda})(I-\frac{\beta\Lambda}{(\beta+\Lambda)(\mu+\gamma)})^2].$$

- The overall sign of $\frac{dV}{dt}$ is determined by the factor of -2 outside the brackets.
- Furthermore, the only point that will make this equation equal to zero is the equilibrium point $(x, y) = (0, 0)$, which corresponds to the values $S = \frac{\Lambda^2}{\mu(\beta+\Lambda)}$ and $I = \frac{\beta\Lambda}{(\beta+\Lambda)(\mu+\gamma)}$.

Global Asymptotic Stability

- So, $\frac{dV}{dt}$ is negative definite in $\mathbb{R}^2$.
- This implies that $\frac{dV}{dt}$ is negative definite on our specified interval D.
- Therefore, our system is **globally asymptotically stable**.