Section 4.3 The Graphs of Polynomial Functions

We first encountered polynomial functions in Section 3.1 where we learned to distinguish between polynomial, rational, and root functions and to identify their domains.

Objective 1: Understanding the Definition of a Polynomial Function

Definition: The function $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$ is a **polynomial function of degree *n*** where n is a nonnegative integer. The numbers $a_0, a_1, a_2, \ldots, a_n$ are called the **coefficients** of the polynomial function. The number a_n is called **the leading coefficient** and a_0 is called the **constant term**.

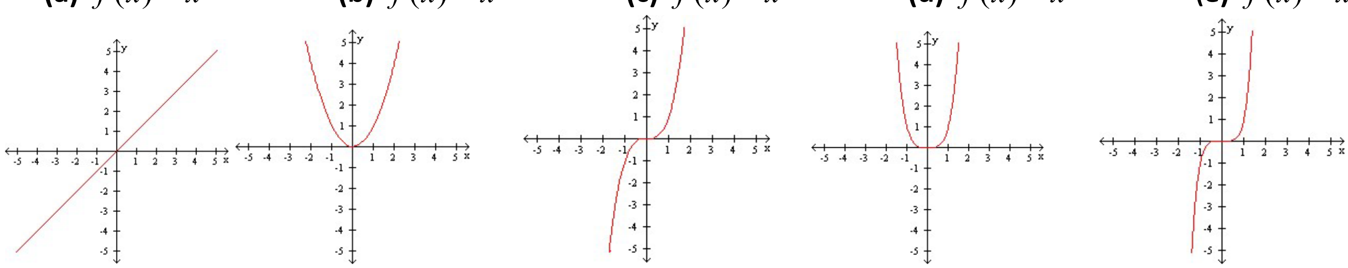
Review of Graphing Transformations of the Square Function and the Cube Function

In section 3.4, we learned how to use transformations to graph families of functions by starting with the graph of a basic function. Two of the basic functions introduced in section 3.3 were the square function, $f(x) = x^2$, and the cube function, $f(x) = x^3$. Both are examples of simple polynomial functions.

Objective 2: Sketching the Graphs of Power Functions

The most basic polynomial functions are functions of the form $f(x) = ax^n$ where n is a positive integer. Functions of this form are called **power functions**. The graphs of five power functions with $a = 1$ are shown below:

(a) $f(x) = x$ **(b)** $f(x) = x^2$ **(c)** $f(x) = x^3$ **(d)** $f(x) = x^4$ **(e)** $f(x) = x^5$



In general if n is even, the graph of $f(x) = x^n$ will resemble the graph of $f(x) = x^2$, and if n is odd, the graph of $f(x) = x^n$ will resemble the graph of $f(x) = x^3$.

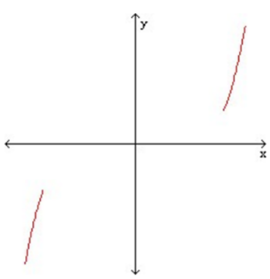
Because we know the basic shape of $f(x) = x^n$, we can use the transformation techniques we have previously learned to sketch the graphs of more complicated polynomial functions.

Objective 3: Determining the End Behavior of Polynomial Functions

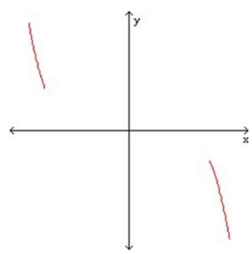
Process for Determining the End Behavior of a Polynomial Function

$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x + a_0$.

If the degree n is odd, the graph has opposite left-hand and right-hand end behavior; that is, the graph "starts" and "finishes" in **opposite** directions.

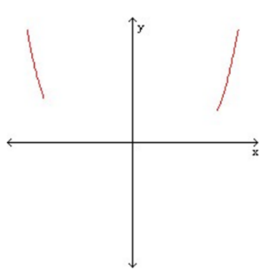


$a_n > 0$, odd degree

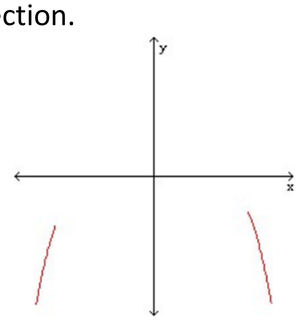


$a_n < 0$, odd degree

If the degree n is even, the graph has the same left-hand and right-hand end behavior; that is, the graph "starts" and "finishes" in the **same** direction.



$a_n > 0$, even degree



$a_n < 0$, even degree

When a polynomial is given in factored form, the sign of the leading coefficient and the degree of the polynomial are not obvious, and it is usually not convenient, nor is it necessary, to multiply the polynomial out completely to get it in the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$ where they are visible. We need only determine the term $a_n x^n$. Here is an example:

Let $f(x) = -7x^4(x^2-5)^6(1-4x)^3$.

First, we notice that $-7x^4$ is a factor of $a_n x^n$.

Next, consider $(x^2-5)^6$. If we were to raise x^2-5 to the 6th power, the largest power of x we would obtain is $(x^2)^6 = x^{12}$. The coefficient of x^{12} would be 1 since the coefficient of x^2 is 1 and $(1)^6 = 1$. Therefore, $(x^2-5)^6$ contributes a factor of x^{12} to the term $a_n x^n$.

Finally, consider $(1-4x)^3$. If we were to raise $1-4x$ to the 3rd power, the largest power of x we would obtain is x^3. The coefficient of x^3 would be $(-4)^3 = -64$. Therefore, $(1-4x)^3$ contributes a factor of $-64x^3$ to the term $a_n x^n$.

Multiplying the three factors we have found, we see that $a_n x^n = -7x^4(x^{12})(-64x^3) = 448x^{19}$. This means the leading coefficient of *f* is positive and the degree of *f* is odd.

Review of Solving Polynomial Equations by Factoring
In section 1.4, we learned how to solve quadratic equations by factoring. In section 1.6, we learned how to solve higher order polynomial equations by factoring.

Review of Evaluating Polynomial Functions for Given Inputs
See section 3.1.

Objective 4: Determining the Intercepts of a Polynomial Function

Every polynomial has a y-intercept that can be found by evaluating $f(0)$.

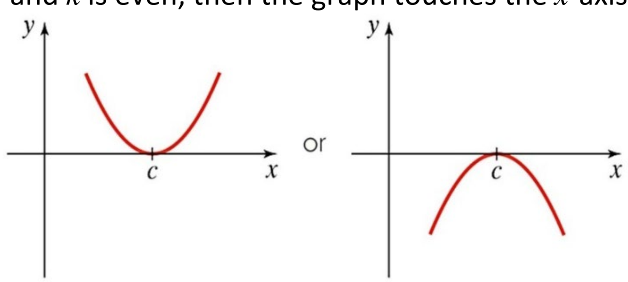
The number $x = c$ is called a **zero** of a function *f* if $f(c) = 0$. If c is a real number, then c is an x-intercept. Therefore, to find the x-intercepts of a polynomial function $y = f(x)$, we must find the real solutions of the equation $f(x) = 0$.

Objective 5: Determining the Real Zeros of Polynomial Functions and Their Multiplicities

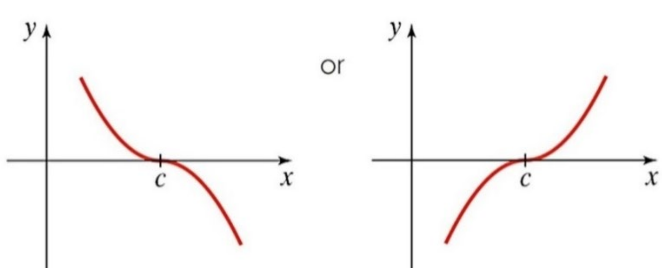
The Shape of the Graph of a Polynomial Function Near a Zero of Multiplicity *k*.

Suppose c is a real zero of a polynomial function *f* of multiplicity k, that is, $(x-c)^k$ is a factor of *f*. Then the shape of the graph of *f* near c is as follows:

If $k > 1$ and k is even, then the graph touches the x-axis at c.



If $k \geq 1$ and k is odd, then the graph crosses the x-axis at c.



Objective 6: Sketching the Graph of a Polynomial Function

Four-Step Process for Sketching the Graph of a Polynomial Function

1. Determine the end behavior.
2. Plot the y-intercept $f(0) = a_0$.
3. Completely factor *f* to find all real zeros and their multiplicities*.
4. Choose a test value between each real zero and sketch the graph.

Remember, without calculus, there is no way to precisely determine the exact coordinates of any "turning points" (also known as relative maximums and relative minimums), on the graph of a polynomial function.

*This is the most difficult step and is discussed in further detail in subsequent sections of the etext.

Objective 7: Determining a Possible Equation of a Polynomial Function Given its Graph