Section 4.6 Rational Functions and Their Graphs

We first encountered polynomial functions in Section 3.1 where we learned to distinguish between polynomial, rational, and root functions and to identify their domains.

Definition: A **rational function** is a function of the form $f(x)=\frac{g(x)}{h(x)}$ where g and h are polynomial functions such that $g(x)$ is any polynomial expression except 0 and the degree of $h(x)$ is greater than zero. If $h(x)=c$ where c is a real number not equal to zero, then we consider the function $f(x)=\frac{g(x)}{h(x)}=\frac{g(x)}{c}$ to be a polynomial.

Review of Simplifying Rational Expressions
Any numeric value of the variable that causes the denominator of a rational expression to equal zero is called a **restricted value**.

One way to simplify a rational expression is to factor the polynomials in the numerator and denominator and divide out any common factors. The restricted values from the original rational expression are still restricted values for the simplified expression.

Review of Finding Intercepts from an Equation
See Section 2.1.

Review of Evaluating Rational Functions for Given Inputs
See Section 3.1

Objective 1: Finding the Domain and Intercepts of Rational Functions

The domain of a rational function is the set of all real numbers x such that $h(x)\neq 0$.

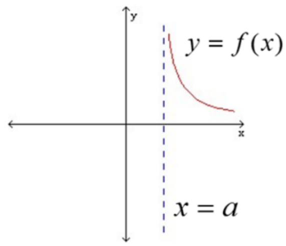
If $f(x)$ has a y-intercept, it can be found by evaluating $f(0)$ provided that $f(0)$ is defined.
If $f(x)$ has any x-intercepts, they can be found by solving the equation $g(x)=0$ (provided that *g* and *h* do not share a common factor).

Objective 2: Identifying Vertical Asymptotes

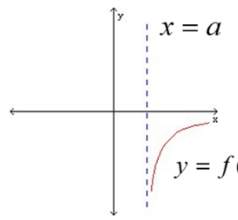
Definition: The vertical line $x = a$ is a **vertical asymptote** of a function $y = f(x)$ if *at least* one of the following occurs:

1. $f(x) \to \infty \text{ as } x \to a^+$
2. $f(x) \to -\infty \text{ as } x \to a^+$
3. $f(x) \to \infty \text{ as } x \to a^-$
4. $f(x) \to -\infty \text{ as } x \to a^-$

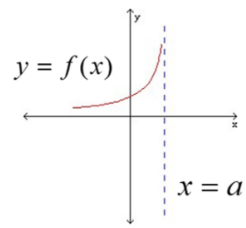
The figures below illustrate each of these cases.



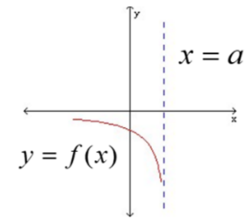
$f(x) \to \infty \text{ as } x \to a^+$



$f(x) \to -\infty \text{ as } x \to a^+$



$f(x) \to \infty \text{ as } x \to a^-$



$f(x) \to -\infty \text{ as } x \to a^-$

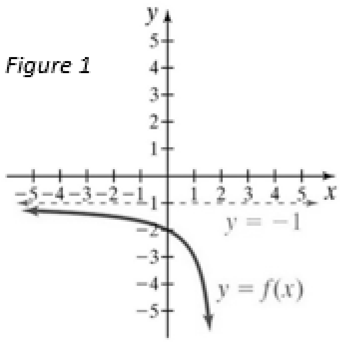
A rational function of the form $f(x) = \dfrac{g(x)}{h(x)}$ where $g(x)$ and $h(x)$ have no common factors will have a vertical asymptote at $x = a$ if $h(a) = 0$.

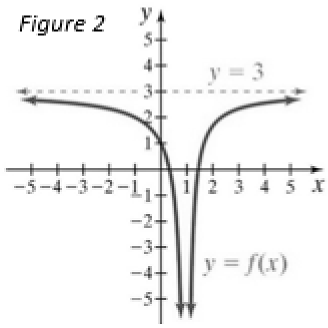
CAUTION **It is essential to divide out any common factors before locating the vertical asymptotes**.

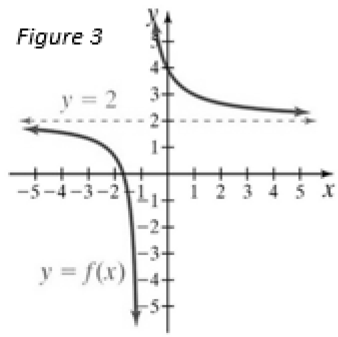
CAUTION **If there is an x-intercept near the vertical asymptote, it is essential to choose a test value that is between the x-intercept and the vertical asymptote.**

Objective 3: Identifying Horizontal Asymptotes

Definition: A horizontal line $y = H$ is a **horizontal asymptote** of a function *f* if the values of $f(x)$ approach some fixed number H as the values of x approach ∞ or $-\infty$.







In Figure 1 above on the left, the line $y = -1$ is a horizontal asymptote because the values of $f(x)$ approach -1 as x approaches $-\infty$.
In Figure 2 above in the middle, the line $y = 3$ is a horizontal asymptote because the values of $f(x)$ approach 3 as x approaches $\pm\infty$.
In Figure 3 above on the right, the line $y = 2$ is a horizontal asymptote because the values of $f(x)$ approach 2 as x approaches $\pm\infty$.

Properties of Horizontal Asymptotes of Rational Functions

- Although a rational function can have many vertical asymptotes, it can have at most one horizontal asymptote.
- The graph of a rational function will never intersect a vertical asymptote but may intersect a horizontal asymptote.
- A rational function $f(x) = \dfrac{g(x)}{h(x)}$ that is written in lowest terms (all common factors of the numerator and denominator have been divided out) will have a horizontal asymptote whenever the degree of $h(x)$ is greater than or equal to the degree of $g(x)$.

Finding Horizontal Asymptotes of a Rational Function

Let $f(x) = \dfrac{g(x)}{h(x)} = \dfrac{a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + b_{m-2}x^{m-2} + \cdots + b_1 x + b_0}$, $a_n \neq 0,\ b_m \neq 0$ where *f* is written in lowest terms, *n* is the degree of *g*, and *m* is the degree of *h*.

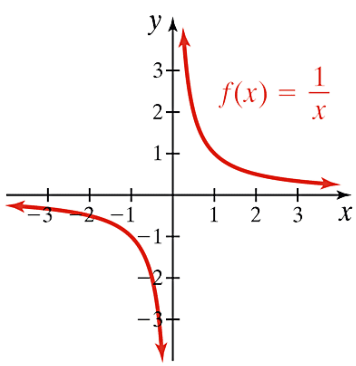
- If $m > n$, then $y = 0$ is the horizontal asymptote.
- If $m = n$, then the horizontal asymptote is $y = \dfrac{a_n}{b_m}$, the ratio of the leading coefficients.
- If $m < n$, then there are no horizontal asymptotes.

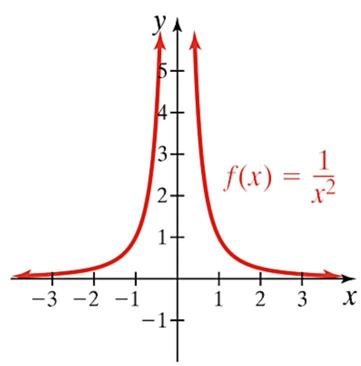
Review of Graphing Transformations of the Reciprocal Function

In section 3.4, we learned how to use transformations to graph families of functions by starting with the graph of a basic function. One of the basic functions introduced in section 3.3 was the reciprocal function, $f(x) = \frac{1}{x}$.

Objective 4: Using Transformations to Sketch the Graphs of Rational Functions

The graphs of $f(x) = \frac{1}{x}$ and $f(x) = \frac{1}{x^2}$





Properties of the graphs of $f(x) = \frac{1}{x}$ and $f(x) = \frac{1}{x^2}$

1. Domain: $(-\infty, 0) \cup (0, \infty)$
2. Range of $f(x) = \frac{1}{x}$: $(-\infty, 0) \cup (0, \infty)$

   Range of $f(x) = \frac{1}{x^2}$: $(0, \infty)$
3. No intercepts
4. Vertical Asymptote: $x = 0$
5. Horizontal Asymptote: $y = 0$
6. $f(x) = \frac{1}{x}$ is an odd function. Its graph is symmetric about the origin and $f(-x) = -f(x)$.

   $f(x) = \frac{1}{x^2}$ is an even function. Its graph is symmetric about the *y*-axis and $f(-x) = f(x)$.

Objective 5: Sketching Rational Functions Having Removable Discontinuities

A rational function $f(x)=\dfrac{g(x)}{h(x)}$ may sometimes have a "hole" in its graph. In calculus, these "holes" are called **removable discontinuities.** Removable discontinuities occur when $g(x)$ and $h(x)$ share a common factor.

Review of Determining if a Function is Even, Odd, or Neither

Knowing if a function is even, odd, or neither can help us to graph it. In section 3.2, we learned how to determine if a function is even, odd, or neither from its equation.

A function *f* is **even** if for every *x* in the domain, $f(-x)=f(x)$. Even functions are symmetric about the *y*-axis. For each point (x,y) on the graph, the point $(-x,y)$ is also on the graph.

A function *f* is **odd** if for every *x* in the domain, $f(-x)=-f(x)$. Odd functions are symmetric about the origin. For each point (x,y) on the graph, the point $(-x,-y)$ is also on the graph.

Objective 7: Sketching Rational Functions

Steps for Graphing Rational Functions of the Form $f(x)=\dfrac{g(x)}{h(x)}$

1. Find the domain.
2. If $g(x)$ and $h(x)$ have common factors, divide out all common factors, determine the coordinates of any removable discontinuities, and rewrite *f* in lowest terms.
3. Check for symmetry.
   If $f(-x)=-f(x)$, then the graph of $f(x)$ is *odd* and thus symmetric about the origin.
   If $f(-x)=f(x)$, then the graph of $f(x)$ is *even* and thus symmetric about the y-axis.
4. Find the y-intercept, if any, by evaluating $f(0)$.
5. Find the x-intercept(s), if any, by finding the zeros of the numerator of *f*, being careful to use the new numerator if a common factor has been removed.
6. Find the vertical asymptotes by finding the zeros of the denominator of *f*, being careful to use the new denominator if a common factor has been removed. Use test values to determine the behavior of the graph on each side of the vertical asymptotes.
7. Determine if the graph has any horizontal asymptotes.
8. Plot points, choosing values of x between each intercept and choosing values of x on either side of the all vertical asymptotes.
9. Complete the sketch.