Section 5.1a Exponential Functions

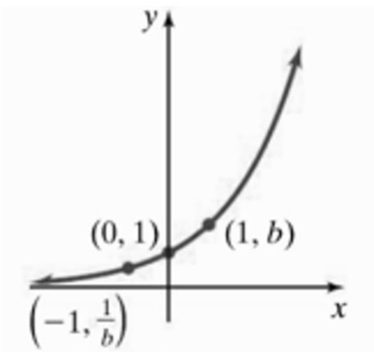
Objective 1: Understanding the Characteristics of Exponential Functions

Definition: An **exponential function** is a function of the form $f(x) = b^x$ where x is any real number and $b > 0$ such that $b \neq 1$. The constant, b, is called the base of the exponential function.

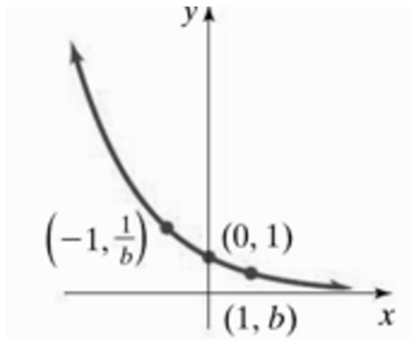
Characteristics of Exponential Functions

For $b > 0$, $b \neq 1$, the exponential function with base b is defined by $f(x) = b^x$.

The domain of $f(x) = b^x$ is $(-\infty, \infty)$ and the range is $(0, \infty)$. The graph of $f(x) = b^x$ has one of the following two shapes depending on the value of *b*:



$f(x) = b^x$, $b > 1$



$f(x) = b^x$, $0 < b < 1$

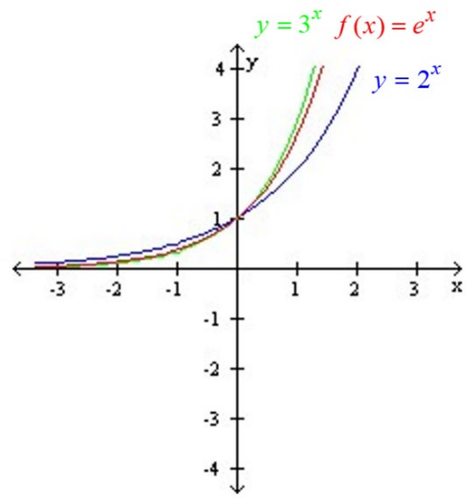
The graph of $f(x) = b^x$, $b > 0$, $b \neq 1$, has the following properties:

1. The graph intersects the y-axis at $(0,1)$.
2. The graph contains the points $\left(-1, \frac{1}{b}\right)$ and $(1, b)$.
3. If $b > 1$, then $b^x \to \infty$ as $x \to \infty$ and $b^x \to 0$ as $x \to -\infty$.
   If $0 < b < 1$, then $b^x \to 0$ as $x \to \infty$ and $b^x \to \infty$ as $x \to -\infty$.
4. The x-axis ($y = 0$) is a horizontal asymptote.
5. The function is one-to-one.

The number e is an irrational number that is defined as the value of the expression $\left(1+\frac{1}{n}\right)^n$ as n approaches infinity. The table below on the left below shows the values of the expression $\left(1+\frac{1}{n}\right)^n$ for increasingly large values of n. As the values of n get large, the value e (rounded to 6 decimal places) is 2.718281.

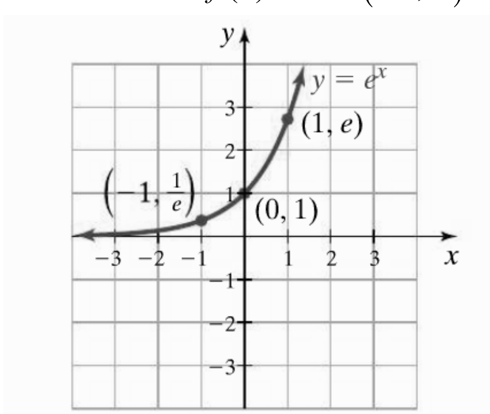
The function $f(x)=e^x$ is called the **natural exponential function.** The graph below on the right shows that the graph of $f(x)=e^x$ lies between the graphs of $f(x)=2^x$ and $f(x)=3^x$ when graphed on the same coordinate system.

n	$\left(1+\frac{1}{n}\right)^n$
1	2
2	2.25
10	2.5937424601
100	2.7048138294
1000	2.7169239322
10,000	2.7181459268
100,000	2.7182682372
1,000,000	2.7182804693
10,000,000	2.7182816925
100,000,000	2.7182818149



Characteristics of the Natural Exponential Function

The Natural Exponential Function is the exponential function with base *e* and is defined as $f(x)=e^x$. The domain of $f(x)=e^x$ is $(-\infty,\infty)$ and the range is $(0,\infty)$.



The graph of $f(x)=e^x$ intersects the y-axis at $(0,1)$.

The graph contains the points $\left(-1,\dfrac{1}{e}\right)$ and $(1,e)$.

$e^x \to \infty$ as $x \to \infty$ and $e^x \to 0$ as $x \to -\infty$.

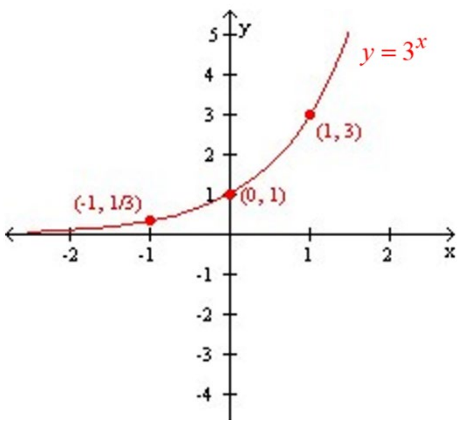
The line $y=0$ is a horizontal asymptote.

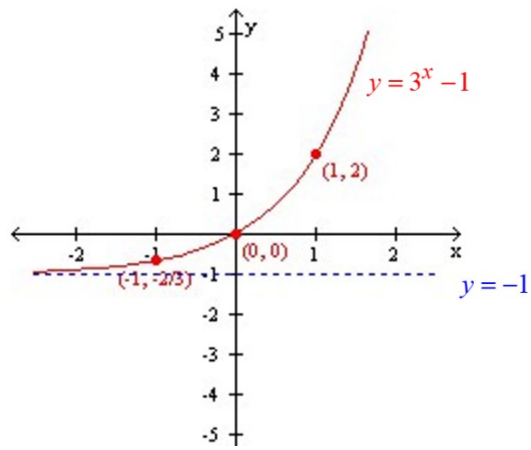
The function $f(x)=e^x$ is one-to-one.

Review of Using Transformations to Graph Functions
See Section 3.4.

Objective 2: Sketching the Graphs of Exponential Functions Using Transformations

The graph of $f(x)=3^x-1$ can be obtained by vertically shifting the graph of $f(x)=3^x$ down one unit. The function $f(x)=3^x$ is graphed below on the left. It contains the points $\left(-1,\frac{1}{3}\right)$, $(0,1)$ and $(1,3)$ and has horizontal asymptote $y=0$. To shift the graph of this function down one unit, subtract 1 from each of the y-coordinates of the points on the graph. The resulting graph of $f(x)=3^x-1$, shown below on the right, contains the points $\left(-1,-\frac{2}{3}\right)$, $(0,0)$ and $(1,2)$ and has horizontal asymptote $y=-1$. The domain of $f(x)=3^x-1$ is $(-\infty,\infty)$ and the range is $(-1,\infty)$.





Review of Properties of Exponents
LSU Video "Exponents" *is available on the course website.*

Here is the list of properties of exponents first introduced in the notes for Section 1.1a.

Product Rule for Exponents
If m and n are positive integers and a is a real number, then

$$a^m \cdot a^n = a^{m+n}.$$

Power Rule for Exponents
If m and n are positive integers and a is a real number, then

$$\left(a^m\right)^n = a^{mn}.$$

Power of a Product Rule
If n is a positive integer and a and b are real numbers, then

$$(ab)^n = a^n \cdot b^n.$$

Power of a Quotient Rule
If n is a positive integer, a and b are real numbers, and $b \neq 0$, then

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

Quotient Rule for Exponents
If m and n are positive integers, a is a real number, and $a \neq 0$, then

$$\frac{a^m}{a^n} = a^{m-n}.$$

Zero Exponent Rule
If b is a real number such that $b \neq 0$, then $b^0 = 1$.

Review of Evaluating Expressions with Negative Exponents
LSU Video "Negative Exponents" *is available on the course website.*
If a is a real number other than 0 and n is an integer, then

$$a^{-n} = \frac{1}{a^n}.$$

Review of Rational Exponents
LSU video "Rational Exponents" *(0:00 – 16:30) is available on the course website.*

Definition of $a^{\frac{1}{n}}$: If n is an integer greater than 1 and $\sqrt[n]{a}$ is a real number, then $a^{\frac{1}{n}} = \sqrt[n]{a}$.

Definition of $a^{\frac{m}{n}}$: If m and n are integers greater than 1 with $\frac{m}{n}$ in lowest terms, then

$a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$ as long as $\sqrt[n]{a}$ is a real number.

Review of Rewriting an Expression in the Form b^u

When solving an equation where the variable is an exponent, it is sometimes useful to rewrite one or both sides of the equation using a different base. For example, 8^x can be rewritten as 2^{3x}.

Objective 3: Solving Exponential Equations by Relating the Bases

The function $f(x) = b^x$ is one-to-one because the graph of *f* passes the horizontal line test. Therefore, if the bases of an exponential equation of the form $b^u = b^v$ are the same, then the exponents must also be the same.

To solve an exponential equation using the **Method of Relating the Bases,** first rewrite the equation in the form $b^u = b^v$. Then $u = v$.

Note that not all exponential equations can be written in the form $b^u = b^v$. Other methods for solving exponential equations will be discussed in Section 5.4.