Section 5.4 Exponential and Logarithmic Equations

Recall the following definitions and properties from earlier in this chapter:

Definition of the Logarithmic Function

For $x > 0,\ b > 0$ and $b \neq 1$, $y = \log_b x$ if and only if $x = b^y$.

The definition of a logarithmic function can be used to rewrite a logarithmic equation as an equation involving an exponent or to rewrite an equation involving an exponent as a logarithmic equation.

Logarithm Property of Equality

If a logarithmic equation can be written in the form $\log_b u = \log_b v$, then $u = v$. Furthermore, if $u = v$, then $\log_b u = \log_b v$.

Properties of logarithms

Let $b > 0,\ b \neq 1$, u and v represent positive numbers, and r be any real number.

1. The **Product Rule for Logarithms** is $\log_b (uv) = \log_b u + \log_b v$.
2. The **Quotient Rule for Logarithms** is $\log_b \frac{u}{v} = \log_b u - \log_b v$.
3. The **Power Rule for Logarithms** is $\log_b u^r = r \log_b u$.

Change of Base Formula

For any positive base $b \neq 1$ and for any positive real number u, then $\log_b u = \frac{\log_a u}{\log_a b}$ where a is any positive number such that $a \neq 1$. Note that the preferred choices for a are usually 10 and e since most calculators are capable of computing expressions containing common and natural logarithms.

Review of Evaluating Expressions with Negative and Rational Exponents

See Section 5.1.

Review of Solving Quadratic Equations by Factoring

See Section 1.4.

Review of Solving Rational Equations

See Section 1.1b.

Review of Solving Exponential Equations by Relating the Bases

Recall from section 5.1 that some exponential equations can be solved by using the **Method of Relating the Bases**. If b is a positive number other than 1 and $b^u = b^v$, then $u = v$.

Objective 1: Solving Exponential Equations

If the equation can be written in the form $\boldsymbol{b^u = b^v}$, then solve the equation $\boldsymbol{u = v}$.

If the equation can be written in the form $\boldsymbol{b^u = c}$ where $\boldsymbol{c}$ is a constant not equal to any power of $\boldsymbol{b}$:

1. Rewrite the equation in logarithmic form using the Definition of a Logarithmic Function.
2. Solve for the given variable and use the Change of Base Formula (base 10 or base *e*) to evaluate.

If the equation cannot be written in the form $\boldsymbol{b^u = b^v}$ or $\boldsymbol{b^u = c}$:

1. Use the Logarithm Property of Equality to "take the log of both sides" (base 10 or base *e*).
2. Use the Power Rule of Logarithms to "bring down" any exponents.
3. Solve for the given variable.

Note that this last method can also be used to solve exponential equations of the form $\boldsymbol{b^u = c}$ where $\boldsymbol{c}$ is a constant not equal to a power of $\boldsymbol{b}$.

Objective 2: Solving Logarithmic Equations

If the equation can be written in the form $\log_b u = \log_b v$, then solve the equation $u = v$.

If the equation cannot be written in the form $\log_b u = \log_b v$:

1. Use Properties of Logarithms to combine all logarithms and write as a single logarithm if needed.
2. Use the Definition of a Logarithmic Function to rewrite the equation in exponential form.
3. Solve for the given variable.
4. Check for any extraneous solutions. Verify that each solution results in the arguments of all logarithms in the original equation being greater than zero.

When solving logarithmic equations, it is important to always verify the solutions. The process of solving logarithmic equations often produces extraneous solutions.