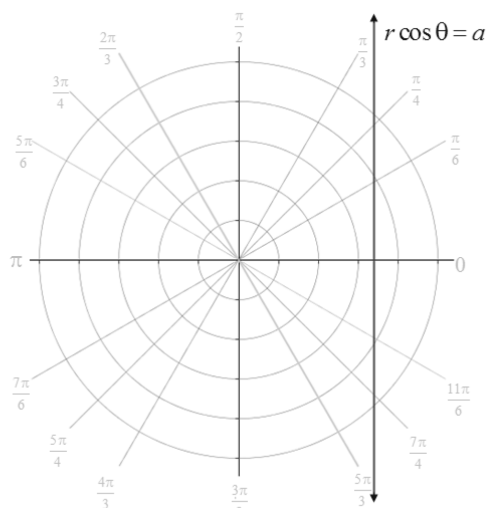
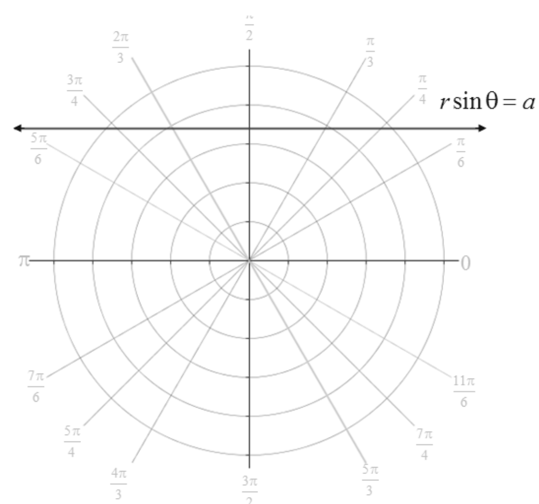
10.2 Graphing Polar Equations

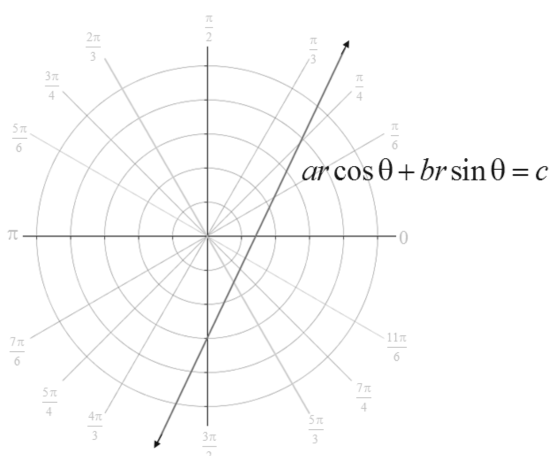
OBJECTIVE 1: Sketching Equations of the Form $r\cos\theta = a$, $r\sin\theta = a$, $ar\cos\theta + br\sin\theta = c$, and $\theta = \alpha$.

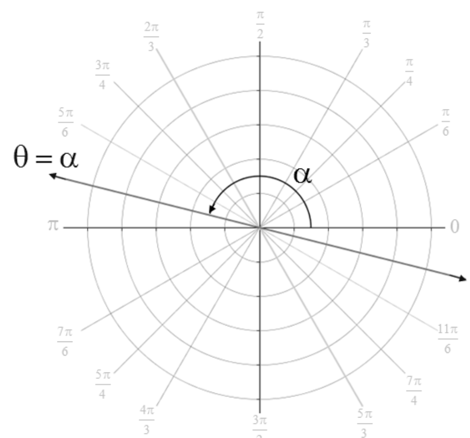
For constants *a, b,* and *c* and angle α, these graphs are **lines.**

- The graph of $r\cos\theta = a$ is a vertical line.
- The graph of $r\sin\theta = a$ is a horizontal line.
- The graph of $ar\cos\theta + br\sin\theta = c$ is a line with slope $-\dfrac{a}{b}$ and *y*-intercept $\dfrac{c}{b}$.
- The graph of $\theta = \alpha$ is a line through the pole that makes an angle of α with the polar axis.





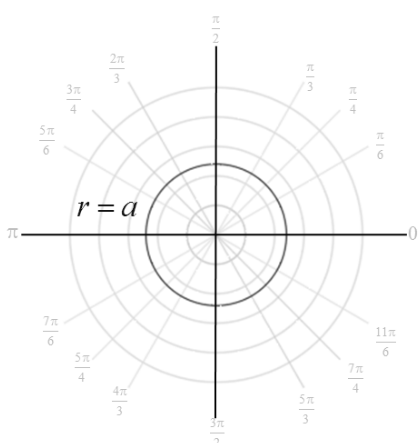


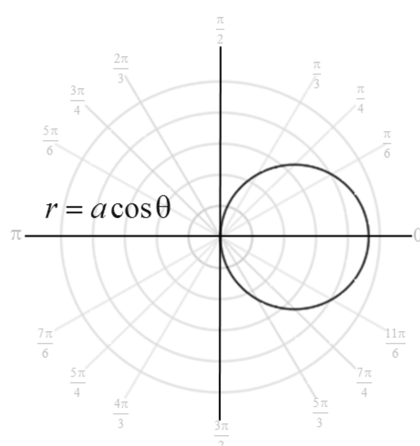


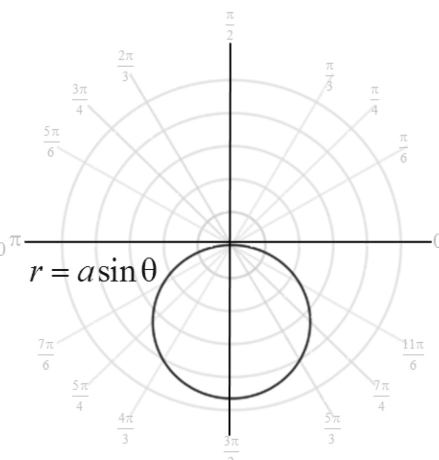
OBJECTIVE 2: Sketching Equations of the Form $r = a$, $r = a\sin\theta$, and $r = a\cos\theta$

For a constant $a \neq 0$, these graphs are **circles**.

- The graph of $r = a$ is a circle centered at the pole with radius of length $|a|$.
- The graph of $r = a\cos\theta$ is a circle centered $\frac{|a|}{2}$ units from the pole on the line $\theta = 0$ with a radius of length $\frac{|a|}{2}$. If $a > 0$, the center of the circle is to the right of the pole, and if $a < 0$, the center of the circle is to the left of the pole.
- The graph of $r = a\sin\theta$ is a circle centered $\frac{|a|}{2}$ units from the pole on the line $\theta = \frac{\pi}{2}$ with radius of length $\frac{|a|}{2}$. If $a > 0$, the center of the circle is above the pole, and if $a < 0$, the center of the circle is below the pole.



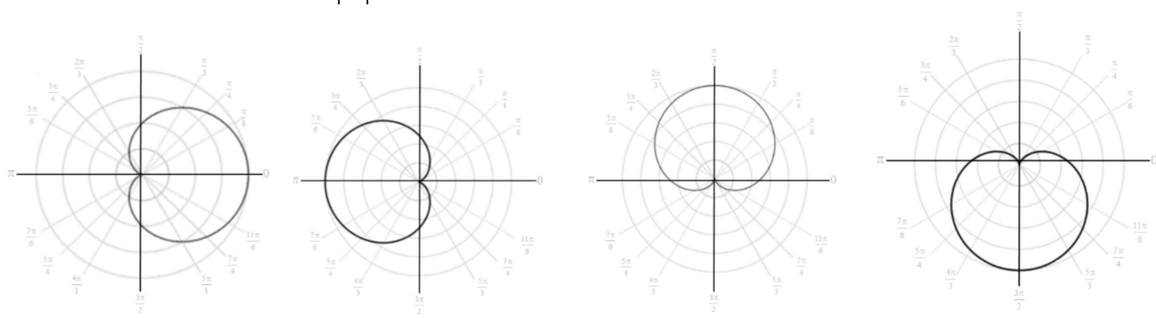




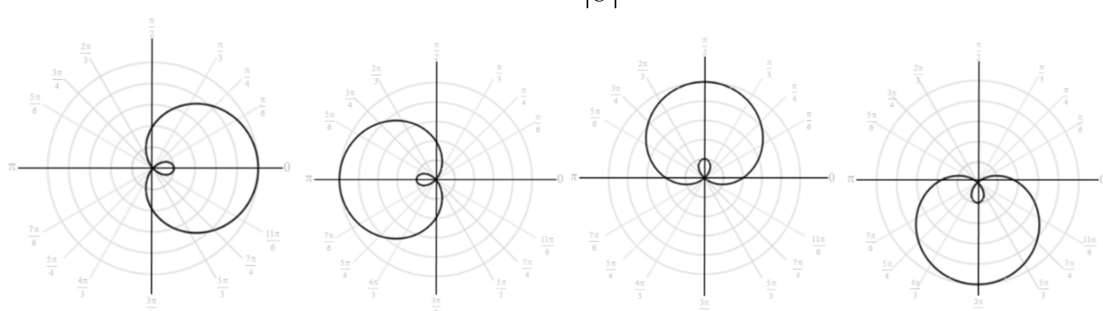
OBJECTIVE 3: Sketching Equations of the Form $r = a + b\sin\theta$ and $r = a + b\cos\theta$

For constants $a \neq 0$ and $b \neq 0$, the shape of the graph of $r = a + b\sin\theta$ and $r = a + b\cos\theta$ is determined by $\left|\frac{a}{b}\right|$.

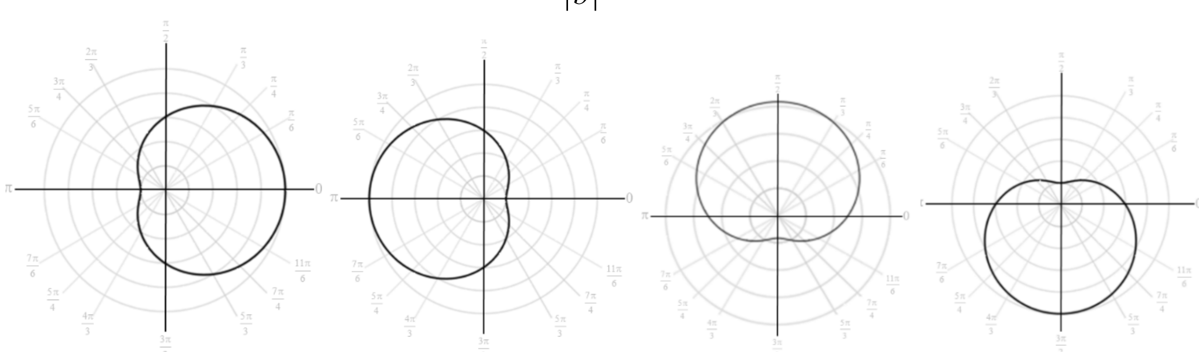
The graph is a **cardioid** if $\left|\frac{a}{b}\right| = 1$.



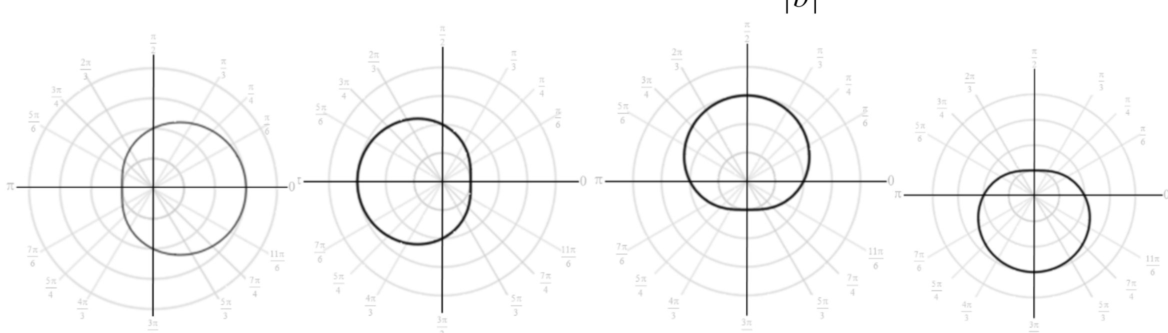
The graph is a **limacon with an inner loop** if $\left|\frac{a}{b}\right| < 1$.



The graph is a **limacon with a dimple** if $1 < \left|\frac{a}{b}\right| < 2$.



The graph is a **limacon with no inner loop and no dimple** if $\left|\frac{a}{b}\right| \geq 2$.



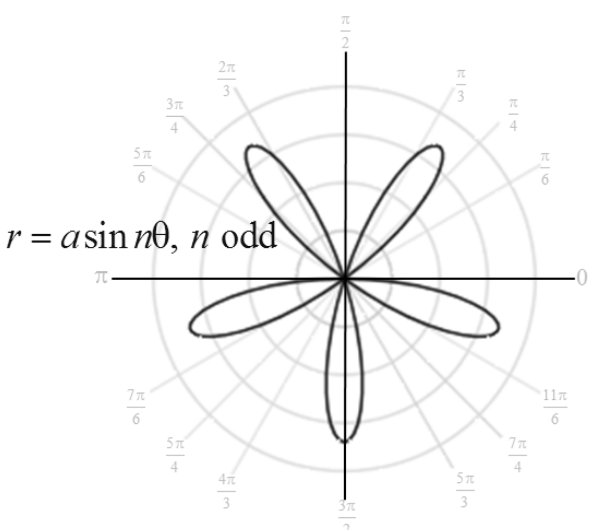
Steps for Sketching Polar Equations (Limacons) of the Form $\boldsymbol{r = a + b\sin\theta}$ and $\boldsymbol{r = a + b\cos\theta}$

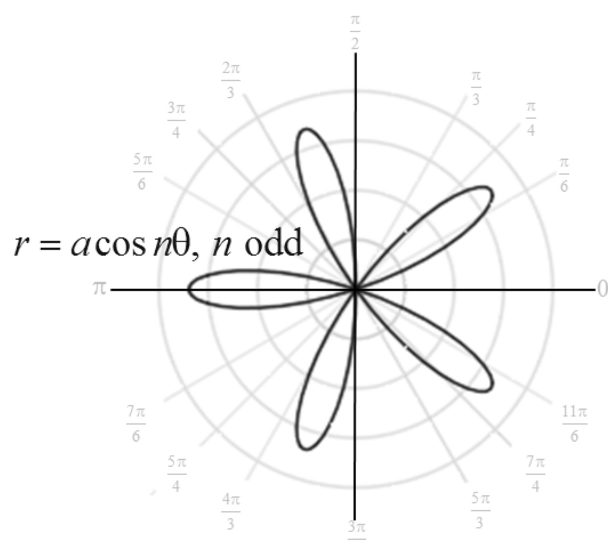
1. Identify the general shape using the ratio $\left|\frac{a}{b}\right|$.
    - If $\left|\frac{a}{b}\right| = 1$, then the graph is a cardioid.
    - If $\left|\frac{a}{b}\right| < 1$, then the graph is a limacon with an inner loop that intersects the pole.
    - If $1 < \left|\frac{a}{b}\right| < 2$, then the graph is a limacon with a dimple .
    - If $\left|\frac{a}{b}\right| \geq 2$, then the graph is a limacon with no inner loop and no dimple.
2. Determine the symmetry.
    - If the equation is of the form $r = a + b\sin\theta$, then the graph must be symmetric about the line $\theta = \frac{\pi}{2}$.
    - If the equation is of the form $r = a + b\cos\theta$, then the graph must be symmetric about the polar axis.
3. Plot the points corresponding to the quadrantal angles $\theta = 0$, $\theta = \frac{\pi}{2}$, $\theta = \pi$, and $\theta = \frac{3\pi}{2}$.
4. If necessary, plot a few more points until symmetry can be used to complete the graph.

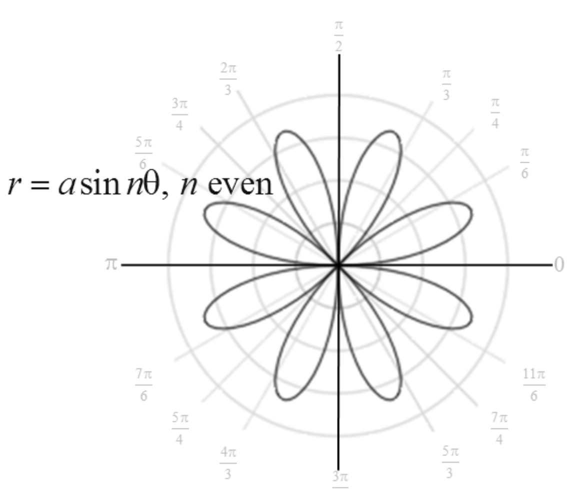
OBJECTIVE 4: Sketching Equations of the Form $r = a\sin n\theta$ and $r = a\cos n\theta$

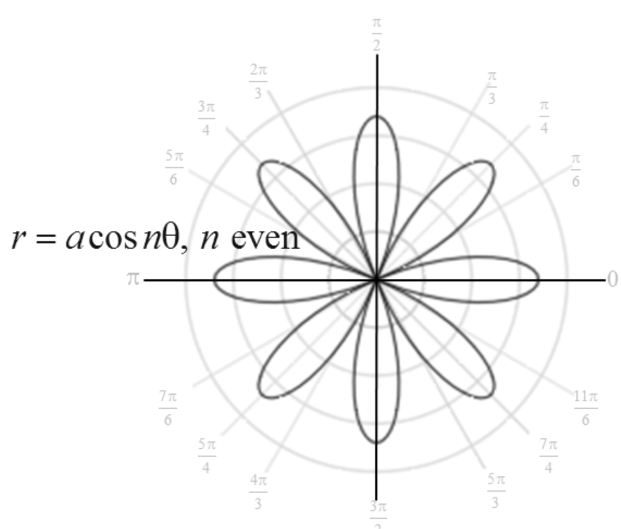
The graphs of polar equations of the form $r = a\sin n\theta$ and $r = a\cos n\theta$ where $a \neq 0$ is a constant and $n \neq 1$ is a positive integer are **roses**.

- The graph of $r = a\sin n\theta$ where *n* is odd is a rose with *n* petals. The endpoint of one petal lies along the vertical line $\theta = \dfrac{\pi}{2}$.
- The graph of $r = a\cos n\theta$ where *n* is odd is a rose with *n* petals. The endpoint of one petal lies along the polar axis.
- The graph of $r = a\sin n\theta$ where *n* is even is a rose with 2*n* petals. None of the petals has an endpoint lying on either the polar axis or the line $\theta = \dfrac{\pi}{2}$.
- The graph of $r = a\cos n\theta$ where *n* is even is a rose with 2*n* petals. Two of the petals have endpoints lying on the line $\theta = \dfrac{\pi}{2}$, and two of the petals have endpoints lying on the polar axis.









Steps for Sketching Polar Equations (Roses) of the Form $r = a\sin n\theta$ and $r = a\cos n\theta$ where $a \neq 0$ and $n \neq 1$ is a positive integer.

1. Identify the number of "petals".
   - If *n* is even, then there are 2*n* petals.
   - If *n* is odd, then there are *n* petals.
2. Determine the length of each petal.
   - The length of each petal is $|a|$ units.
3. Determine all angles where an endpoint of a petal lies.
   - If the equation is of the form $r = a\sin n\theta$, then the endpoints occur for angles on the interval $[0, 2\pi)$* that satisfy the equations $\sin n\theta = 1$ and $\sin n\theta = -1$.
   - If the equation is of the form $r = a\cos n\theta$, then the endpoints occur for angles on the interval $[0, 2\pi)$* that satisfy the equations $\cos n\theta = 1$ and $\cos n\theta = -1$.

   *Note that when *n* is odd, it is only necessary to consider angles on the interval $[0, \pi)$. A complete graph is obtained on this interval because the graph will completely traverse itself on the interval $[\pi, 2\pi)$.
4. Substitute each angle determined in Step 3 back into the original equation to obtain the appropriate values of *r* for each angle. The ordered pairs obtained represent the endpoints of the rose petals. Plot these points on the graph.
5. Determine angles where the graph passes through the pole. These angles serve as a guide when sketching the width of a petal.
   - If the equation is of the form $r = a\sin n\theta$, then the graph passes through the pole when $\sin n\theta = 0$.
   - If the equation is of the form $r = a\cos n\theta$, then the graph passes through the pole when $\cos n\theta = 0$.

6. Draw each petal to complete the graph.

OBJECTIVE 5: Sketching Equations of the Form $r^2 = a^2 \sin 2\theta$ and $r^2 = a^2 \cos 2\theta$

The graphs of polar equations of the form $r^2 = a^2 \sin 2\theta$ and $r^2 = a^2 \cos 2\theta$ where $a \neq 0$ is a constant are **lemniscates**.

- The graph of $r^2 = a^2 \sin 2\theta$ is a lemniscate symmetric about the pole and the line $\theta = \frac{\pi}{4}$. The endpoints of the two loops occur when $\theta = \frac{\pi}{4}$ and $\theta = \frac{5\pi}{4}$. The length of the loops is $|a|$.
- The graph of $r^2 = a^2 \cos 2\theta$ is a lemniscate symmetric about the pole, the horizontal line $\theta = 0$, and the vertical line $\theta = \frac{\pi}{2}$. The endpoints of the two loops occur when $\theta = 0$ and $\theta = \pi$. The length of the loops is $|a|$.